

Monochromatic cycle partitions of 2-coloured graphs with minimum degree $3n/4$

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Abstract

Balogh, Barát, Gerbner, Gyárfás, and Sárközy proposed the following conjecture. Let G be a graph on n vertices with minimum degree at least $3n/4$. Then for every 2-edge-colouring of G , the vertex set $V(G)$ may be partitioned into two vertex-disjoint cycles, one of each colour.

We prove that this conjecture holds for n large enough, improving approximate results by the aforementioned authors and by DeBiasio and Nelsen.

1 Introduction

1.1 History

While undergraduates in Budapest, Gerencsér and Gyárfás [14] proved the following simple result: for any 2-edge-colouring of the complete graph K_n , there exists a monochromatic path of length at least $\lceil 2n/3 \rceil$. It is easy to see that this statement is sharp. In their paper, Gerencsér and Gyárfás observe that a weaker result, asserting the existence of a monochromatic path of length at least $n/2$, can be deduced from the following simple observation: for any red and blue colouring of K_n , there is a Hamilton path which is the union of a red path and a blue path. The latter observation, simple as it is, inspired intensive research.

In a later paper, Gyárfás [12] proved that, in fact, more is true. He showed that for any red and blue colouring of K_n the vertices may be covered by a red cycle and a blue one sharing at most one vertex. Lehel went even further: he conjectured that for every 2-colouring of K_n the vertex set may be partitioned into two monochromatic cycles of distinct colours. This conjecture first appeared in [2], where it was proved for some special colourings of K_n .

Almost twenty years after this conjecture was made, Łuczak, Rödl and Szemerédi [22] proved it for large n , using the Regularity Lemma. Ten years later, Allen [1] proved it for large n avoiding the use of the Regularity Lemma. Finally, Lehel's conjecture was fully resolved by Bessy and Thomassé [5] with an elegant and short proof.

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1.2 Conjectures and progress

In the hope of generalising the above result of Gerencsér and Gyárfás, Schelp [26] considered 2-colourings of graphs which are not necessarily complete. In particular, he conjectured that given a graph G on n vertices with $\delta(G) > 3n/4$, there is a monochromatic path of length at least $2n/3$. Benevides, Luczak, Skokan, Scott and White [4] and Gyárfás and Sárközy [17] proved approximate versions of this conjecture.

Inspired by Schelp's conjecture, Balogh, Barát, Gerbner, Gyárfás, and Sárközy [3] proposed the following conjecture: given a graph G on n vertices with minimum degree $\delta(G) > 3n/4$, for every 2-colouring of G the vertex set can be partitioned into two monochromatic cycles of distinct colours. We remark that for the purpose of this conjecture, the empty set, a single vertex and an edge are considered to be cycles. We note that there are examples of 2-coloured graphs G on n vertices with $\delta(G) = \lceil 3n/4 - 1 \rceil$ which do not admit such a partition (see Section 2).

In [3], the authors prove the following approximate result of their conjecture. For every $\varepsilon > 0$ there exists n_0 such that for every 2-coloured graph G on $n \geq n_0$ vertices with minimum degree $\delta(G) \geq (3/4 + \varepsilon)n$, there exist vertex-disjoint monochromatic cycles of distinct colours covering all but at most εn of the vertices.

Recently, DeBiasio and Nelsen [8] proved the following stronger approximate result of the latter conjecture: for every $\varepsilon > 0$ there exists n_0 such that, for every 2-coloured graph G on $n \geq n_0$ vertices and $\delta(G) \geq (3/4 + \varepsilon)n$, the vertex set may be partitioned into two monochromatic cycles of distinct colours.

1.3 The main result

Our main aim is to prove that the conjecture of Balogh et al. [3] holds if n is large enough.

Theorem 1.1. *There exists n_0 such that if a graph on $n \geq n_0$ vertices and minimum degree at least $3n/4$ is 2-coloured then its vertex set may be partitioned into two monochromatic cycles of different colours.*

In [21], Luczak introduced a technique that uses the Regularity Lemma to reduce problems about paths and cycles into problems about connected matchings, which are matchings that are contained in a connected component. This technique, which we shall describe in more detail in Section 6, has become fairly standard by now and can be used to prove the approximate result of Balogh et al. [3]. The second result by DeBiasio and Nelsen [8], requires further ideas, most notably the “absorbing technique” of Rödl, Ruciński and Szemerédi (see [25] and [20]). Nevertheless, the stronger conditions on the minimum degree make their proof a great deal easier than ours. In order to prove Theorem 1.1, we use a variety of additional ideas and techniques.

We remark that Theorem 1.1 is sharp. Indeed, for every $n \geq 4$, there exists a 2-coloured graph on n vertices with minimum degree $\lceil \frac{3n}{4} - 1 \rceil$ admitting no partition into two monochromatic cycles of distinct colours. We give such extremal examples in the following section. These examples disprove the conjecture from [8], that a slightly stronger version of Theorem 1.1 may hold, namely that the conclusion holds for graphs with minimum degree at least $\frac{3n-3}{4}$.

The following section consists of some extremal examples for Theorem 1.1. In Section 3 we give an overview of the proof as well as the structure of the rest of this paper.

2 Sharpness examples

Before we turn to the proof of Theorem 1.1, we give some extremal examples showing that the theorem is sharp. More precisely, we give examples of 2-coloured graphs on n vertices with minimum degree $\lceil \frac{3n}{4} - 1 \rceil$ admitting no partition into two monochromatic cycles of distinct colours. Figures (1, 2, 3, 4) depict several families of such examples differing in the values of $n \pmod{4}$.

In these figures, we use black and grey for the colours of the edges. The areas coloured in either black or grey denote a complete (or complete bipartite) subgraph of the corresponding colour. Shaded areas denote complete (or complete bipartite) subgraphs which may be coloured arbitrarily. White areas are empty subgraphs. A small dot denotes a single vertex and a larger shape denote a cluster of vertices whose size is written in it. It is not hard to see that, indeed, each of these figures is an extremal example for Theorem 1.1. We leave the details to the reader.

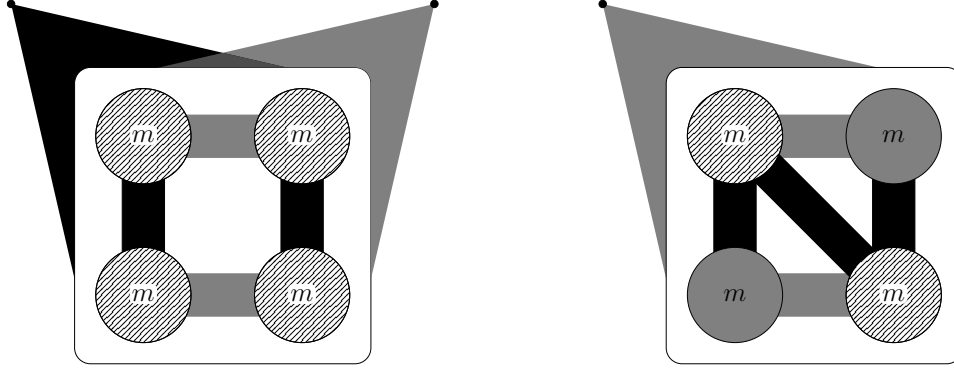


Figure 1: Black and grey graphs of orders $4m + 2$ and $4m + 1$ (from left to right) with minimum degrees $3m + 1$ and $3m$ respectively, admitting no partition into a black cycle and a grey one.

Other extremal examples may be formed by removing a vertex or two from the left-hand graph, or by removing the extra vertex from the right-hand graph.

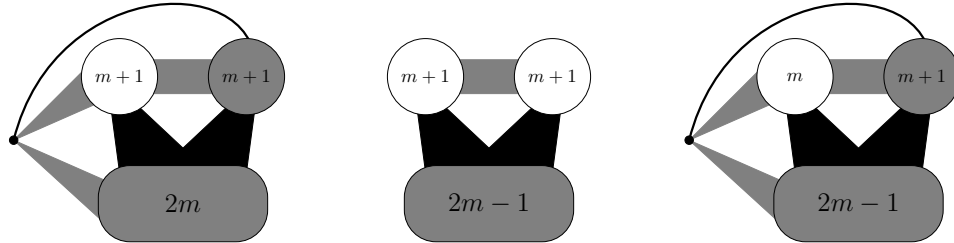


Figure 2: Black and grey graphs of orders $4m$, $4m + 1$ and $4m + 2$ (left to right) with minimum degrees $3m - 1$, $3m$ and $3m + 1$ respectively and no partition into two monochromatic cycles of distinct colours.

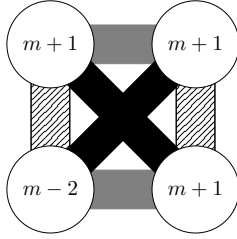


Figure 3: A black and grey graph on $4m + 1$ vertices and minimum degree $3m$ with no partition into a grey cycle and a black one.

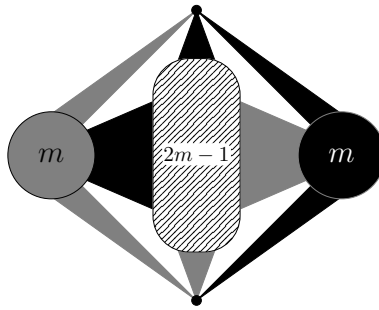


Figure 4: A black and grey graph on $4m + 1$ vertices and minimum degree $3m$ with no partition into a black cycle and a grey one. Any subgraph of this graph, obtained by removing one vertex, is also an extremal example.

We remark that there may exist other extremal examples. It may be possible to find all of them by carefully analysing the proof of Theorem 1.1, but we make no attempt to do so at this point.

3 Outline of the proof and structure of the paper

In this section we describe the proof of our main theorem (1.1). We are given a graph G on n vertices and minimum degree $3n/4$ with a red and blue colouring of the edges. In what follows, we give an outline of our proof that $V(G)$ may be partitioned into a red cycle and a blue one.

Given an edge coloured graph, a *monochromatic connected matching* is a monochromatic matching which is contained in a connected component of the same colour. Similarly to the earlier results on our problem (by Balogh et al. [3] and DeBiasio and Nelsen [8]), as well as many other results in the area, one of the key tools is the technique of reducing problems about cycles to problems about connected matchings using the Regularity Lemma. This technique was introduced by Łuczak [21] and since then has become fairly standard. In our setting, the basic idea, which is described in more detail in Section 6, is as follows. We are given a 2-coloured graph G and consider the reduced graph obtained by applying the Regularity Lemma. If the reduced graph has a perfect matching consisting of a red connected matching and a blue connected matching, we may use the blow-up

lemma [18] (or in fact, a much simpler special case), to find two vertex-disjoint monochromatic cycles, a red one and a blue one, which cover almost all of the vertices.

The next ingredient is the “absorbing method” of Rödl, Ruciński and Szemerédi (see [25] and [20]). As in [8], in order to apply this method, we use a notion of “robust subgraphs”, which are defined to be graphs with certain expansion properties (see Section 5 for the exact definition). Such graphs can be shown to contain short “absorbing paths”, which are paths that can absorb small sets of vertices. We observe (see Section 6) that monochromatic connected components in the reduced graph (obtained from the Regularity Lemma) correspond to monochromatic robust subgraphs in the original graph. This observation allows us to obtain information about the rough structure, by using the Regularity Lemma and finding the corresponding robust subgraphs (see Section 8).

After a study of the some properties of the rough “robust structure” of the graph, we aim to apply the Regularity Lemma again, in order to find a suitable perfect matching, namely a perfect matching consisting of a connected red matching and a connected blue matching. This matching is used to find two vertex-disjoint cycles, a red one and a blue one, which cover most of the vertices and have the additional absorbing property implying that the leftover vertices can be inserted into one of these cycles (see Sections (9 - 14)).

We should like to emphasize that as we prove the sharp result, namely that we only assume that the minimum degree is at least $3n/4$, rather than $(3/4 + \varepsilon)n$ as in the previous results, new difficulties arise, making our proof much harder. Firstly, we have to deal with several different cases for the rough robust structure, some of which do not arise when $\delta(G) \geq (3/4 + \varepsilon)n$. Interestingly, these cases require a variety of ideas and techniques, making the proof of the general theorem rather intricate. Secondly, when applying the Regularity Lemma we cannot guarantee that the minimum degree would be at least $3n/4$, thus it may not be possible to find a suitable matching in the reduced graph directly.

The combination of the following two ideas helps us with these challenges. The first idea is to use stability versions of results promising a perfect matching. These enable us to conclude that if the reduced graph does not have the required perfect matching, it has some specific structure which we can further analyse to find the required monochromatic cycle partition.

The second idea is the following simple yet important observation: given two robust components of the same colour, if they can be connected with two vertex-disjoint paths, they may essentially be treated as one larger component. In several parts of the proof (see Sections (10, 12, 13)), we use this observation to conclude that either we may join two robust components to obtain a larger one, or the graph admits some restrictive structure, for which the desired partition may be found “by hand” (see Sections 15 and 16). We remark that even at this stage, the proof is rather hard due to the fact that our result is sharp.

3.1 Structure of the paper

In the next section, Section 4, we introduce the notation that will be used in this paper. In Section 5, we define our notion of robustness and prove some properties of robust components, most notably the existence of absorbing paths. In Section 6, we state the version of the Regularity Lemma that we use here. We also prove some results about the correspondence between connected

components of the reduced graph and robust subgraphs of the original graph and describe the method of converting connect matchings in the reduced graph into cycles in the original graph. In Section 7, we list some results which will be used throughout the proof.

Sections (8 - 16) are devoted to the proof of Theorem 1.1. In Section 8, we obtain some information about the rough structure and point out how to prove Theorem 1.1 using the results in subsequent lemmas. In each of Sections 9 - 14, we consider one of the cases arising from the structural result in Section 8. These cases vary in difficulty and we have to use various techniques used to deal with them. In Sections 15 and 16, we prove Lemmas 7.13 and 7.14, which are used in earlier sections and prove the main theorem under certain restrictive conditions on the colouring and the structure of the graph. Finally, Section 17 is devoted to some concluding remarks.

4 Notation

We use mostly standard notation. Write $|G|$ for the order of a graph G and $\delta(G)$ and $\Delta(G)$ for its minimum and maximum degrees respectively. The neighbourhood of a vertex $x \in V(G)$ is denoted by $N_G(x)$ and its degree by $d_G(x) = |N_G(x)|$. Given $A \subseteq V(G)$, we write $N_G(x, A) = N_G(x) \cap A$ and $d_G(x, A) = |N_G(x, A)|$. We will write, for example, $d(x, A)$ for $d_G(x, A)$ if this is unambiguous. Given a set of vertices $X \subseteq V(G)$, we write $G[X]$ for the graph induced by G on X . Similarly, for disjoint subsets $X, Y \subseteq V(G)$, we write $G[X, Y]$ for the bipartite graph with bipartition $\{X, Y\}$ induced by G . We denote $e_G(X, Y) = |E(G[X, Y])|$.

Given a graph G , we denote a 2-colouring of G by $E(G) = E(G_B) \cup E(G_R)$, where G_B, G_R are graphs on vertex set $V(G)$ (in proper colouring, the graphs G_B, G_R are edge-disjoint). The edges of G_B are called blue edges and the edges of G_R are red edges. We sometimes use B or R for a subscript instead of G_B or G_R . For instance, $N_B(x)$ is a shorthand for $N_{G_B}(x)$.

We denote by $(u_1 u_2 \dots u_k)$ the path on vertices u_1, \dots, u_k taken in this order. We use the same notation to denote the cycle obtained by adding the edge (u_k, u_1) to the given path. It should be clear from the context if we are dealing with a path or a cycle. Given paths P_1, P_2 which share an end and are otherwise disjoint, we denote by $(P_1 P_2)$ the concatenation of the two paths. Similarly, if the paths share both ends but are otherwise disjoint, the same notation denotes the cycle obtained by joining the two paths.

Throughout this paper we omit floors and ceilings whenever the argument is unaffected. The constants in the hierarchies used to state our results are chosen from right to left. For example, the claim that a statement holds for $0 < a, \frac{1}{n} \ll b \ll c \ll 1$ means that there exist non-decreasing functions $f, g : (0, 1] \rightarrow (0, 1]$ and a constant c_0 such that the statement holds for all $0 < a, b, c \leq c_0$ and integers n with $b \leq f(c), a \leq g(b)$ and $n \geq \frac{1}{g(b)}$. We normally do not specify the functions in question.

5 Robust components and absorbing paths

Similarly to the proof of DeBiasio and Nelsen [8], one of the main tools in our proof is the notion of robust subgraphs. As we shall see, these are graphs with certain expansion properties. The role of robust subgraphs in our proof is similar to their role in [8], but our definition is different

and is often easier to apply. Nevertheless, the two definitions are in some sense equivalent, as can be seen in Lemma 5.4 in [8]. After defining a robust subgraph, we state and prove some simple properties of these components. Finally, we prove that robust subgraphs contain “absorbing paths”, which may absorb small sets of vertices.

5.1 Definitions of robust subgraphs

We define two notions of robustness: strong and weak. The difference between the two is that strong robust subgraphs are far from being bipartite. It will be easier for our application, though not essential, to define a robust subgraph relative to a fixed ground graph. The precise definitions are as follows.

Given a graph G , vertices $x, y \in V(G)$ and an integer l , denote by $\text{con}_{G,l}(x, y)$ the number of paths of length $l + 1$ in G between x and y .

Definition 5.1. Let G be a graph on n vertices.

A subgraph F of G is called (α, k) *strongly robust* if there exists $l \leq k$ such that for every pair of vertices x, y in F we have $|\text{con}_{F,l}(x, y)| \geq \alpha n^l$.

A subgraph F of G is called (α, k) *weakly robust* if there exists a partition $\{X, Y\}$ of $V(F)$ such that for some $l \leq k$ and every $x \in X, y \in Y$ we have $|\text{con}_{F',l}(x, y)| \geq \alpha n^l$, where $F' = F[X, Y]$.

As a shorthand, we often omit the parameters α and k when they are clear from the context. From now on, we use the term “robust” to signify either strongly robust or weakly robust with suitable parameters. We point out that in our context α and k are fixed and n tends to infinity. We remark that an (α, k) robust subgraph F of a graph G on n vertices has $\delta(F) \geq \alpha n$. In particular, robust subgraphs are always dense.

Before discussing some properties of robust subgraphs, let us give a few examples. Any graph of minimum degree at least $(1/2 + \alpha/2)n$ is $(\alpha, 1)$ strongly robust, because any two vertices have at least αn common neighbours. Similarly, the random graph $G(n, \alpha)$, is w.h.p. $(\alpha^2/2, 1)$ strongly robust. Furthermore, the blow-up of a path of length $k \geq 2$, where every vertex is replaced by a complete graph on n/k vertices, is $(\alpha, k - 1)$ strongly robust for a suitable α . Similarly, if vertices of a path of length k are replaced by empty graphs, we obtain an $(\alpha, k - 1)$ weakly robust graphs.

5.2 Properties of robust subgraphs

We shall make use of some simple properties of robust graphs. The following lemma states that a robust subgraph remains robust after removing a small number of vertices.

Lemma 5.2. *Given $\alpha > 0$ and k an integer, the following holds for small enough β . Let G be a graph on n vertices and let F be an (α, k) -robust subgraph. Suppose that F' is obtained from F by removing at most βn vertices. Then F' is $(\alpha/2, k)$ -robust.*

Proof. We prove the lemma under the assumption that F is strongly robust; the proof in case F is weakly robust is analogous. Let $l \leq k$ satisfy $|\text{con}_{F,l}(x, y)| \geq \alpha n^l$ for every $x, y \in V(F)$. For every $x, y \in V(F)$, the number of paths of length $l + 1$ between x and y containing at least one

vertex from $V(F) \setminus V(F')$ is at most $l\beta n^l \leq \frac{\alpha}{2}n^l$. It follows that $|\text{con}_{F',l}(x,y)| \geq \frac{\alpha}{2}n^l$ for every $x,y \in V(F)$, i.e. F' is $(\alpha/2, l)$ strongly robust. \square

The next lemma shows that a robust subgraph remains robust after removing a graph of small maximum degree.

Lemma 5.3. *Given $\alpha > 0$ and k , the following holds for suitably small β and large n . Let G be a graph on n vertices and let F be an (α, k) -robust subgraph. Suppose that F' is a subgraph of F such that for every vertex $v \in V(F)$ we have $\deg_{F'}(v) \geq \deg_F(v) - \beta n$. Then F' is $(\alpha/2, k)$ -robust.*

Proof. We prove this lemma for F strongly robust; the proof for F weakly robust is similar. Let $l \leq k$ be such that $|\text{con}_{F,l}(x,y)| \geq \alpha n^l$ for every $x,y \in V(F)$. Fix some $x,y \in V(F)$. We consider the family of paths in F of length $l+1$ between x and y which contain at least one edge outside of F' . There are at most $l\beta n^l$ such paths, i.e. $|\text{con}_{F',l}(x,y)| \geq |\text{con}_{F,l}(x,y)| - l\beta n^l \geq \frac{\alpha}{2}n^l$ (for small enough β). It follows that F' is $(\alpha/2, l)$ strongly robust. \square

The following lemma states that a robust subgraph F remains robust after the addition of vertices which have a large neighbourhood in F .

Lemma 5.4. *Given $\alpha > 0$ and k an integer, the following holds for large enough n . Let G be a graph on n vertices and let F be an (α, k) -robust component. Let F' be a subgraph of G containing F , such that every vertex in $V(F') \setminus V(F)$ has at least αn neighbours in F . Then F' is $(\alpha^3/2, k+2)$ -robust.*

Proof. We prove the statement assuming that F is strongly robust; the proof in case F' is weakly robust is very similar and we omit the details. Let $l \leq k$ satisfy $|\text{con}_{F,l}(x,y)| \geq \alpha n^l$ for every $x,y \in V(F)$. Fix some $x,y \in V(F')$. For every $z,w \in V(F)$ such that $z \in N(x)$ and $w \in N(y)$, we have $|\text{con}_{F,l}(w,z)| \geq \alpha n^l$. Thus, since every vertex in $V(F')$ has at least αn neighbours in F , we have that the number of walks between x and y in F' with $l+2$ inner vertices is at least $\alpha^3 n^{l+2}$ for large n . Since there are at most $O(n^{l+1})$ such walks which are not paths, we have that $|\text{con}_{F',l}(x,y)| \geq \frac{\alpha^3}{2}n^{l+1}$. It follows that H' is $(\alpha^3/2, k+2)$ -strongly robust. \square

So far we listed and proved several simple properties of robust subgraphs. In the following subsection we state and prove a more interesting property.

5.3 Absorbing paths

The main reason robust subgraphs are so useful in our context, is the fact, which was proved by DeBiasio and Nelsen [8] that they contain short “absorbing paths”. We conclude this section with a proof of this fact.

Lemma 5.5. *Let $\frac{1}{n} \ll \rho \ll \alpha, \frac{1}{k} \ll 1$, let G be a graph on n vertices and let F be an (α, k) -robust subgraph of G . Then there exists a path Q in F satisfying the following conditions.*

1. If F is strongly robust, for every set $W \subseteq V(F) \setminus V(Q)$ of size at most $\rho^2 n$, there exists a Hamilton path in $F[V(Q) \cup W]$ with the same ends as Q .
2. If F is weakly robust with bipartition $\{X, Y\}$, for every $W \subseteq V(F) \setminus V(P)$, with $|W \cap X| = |W \cap Y| \leq \rho^2 n$, the graph $F[V(Q) \cup W]$ contains a Hamilton path with the same ends as Q .

We follow the footsteps of DeBiasio and Nelsen in their proof of Lemma 5.6 from [8]. The main tool is absorbing method of Rödl, Ruciński and Szemerédi [25]. We shall use “gadgets”, which we will define to be Hamiltonian graphs that are can absorb a single vertex under the condition that it is adjacent to some of the vertices in the gadget. By a simple application of the probabilistic method and the robustness of the given graph, we show that there exists a not too large collection of vertex-disjoint gadgets, such that every vertex may be absorbed by a rather large number of them. From there it will be easy to construct the required path Q .

Proof of Lemma 5.5. We start by proving the first part of Lemma 5.5. Suppose that F is (α, k) -strongly robust. In particular, $\delta(F) \geq \alpha n$ so we may apply the following claim.

Claim 5.6. *Let $p, \frac{1}{n} \ll \alpha \ll 1$ and let F be a graph on at most n vertices with $\delta(G) \geq \alpha n$. Then there exists a family \mathcal{F} of disjoint pairs of vertices of $V(F)$ such that the following conditions hold.*

- $|\mathcal{F}| \leq pn$
- For every $u \in V(F)$ there are at least $\frac{1}{16}p\alpha n^2$ pairs $(x, y) \in \mathcal{F}$ such that x, y are neighbours of u .

This claim is a simple application of Chernoff’s bound. We shall pick a family of pairs of vertices randomly and then delete a small number of pairs so as to ensure that the pairs are disjoint.

Proof of Claim 5.6. Let \mathcal{F} be the family of pairs obtained by choosing each pair of vertices in $V(G)$ independently with probability $\frac{p}{n}$. By Chernoff’s bound, we have that with high probability, the following properties hold.

- $|\mathcal{F}| \leq 2\frac{p}{n}\binom{n}{2} \leq pn$.
- For every $u \in V(F)$, \mathcal{F} contains at least $\frac{1}{2}\frac{p}{n}\binom{\alpha n}{2} \geq \frac{1}{8}p\alpha^2 n$ pairs (x, y) such that $x, y \in N(u)$.

The expected number of pairs of intersecting pairs in \mathcal{F} is at most $\left(\frac{p}{n}\right)^2 n^3 \leq p^2 n$. It follows by Markov’s inequality that with probability at least $1/2$, the number of pairs of intersecting pairs in \mathcal{F} is at most $2p^2 n$. In particular, we may pick a family \mathcal{F} which satisfies the above conditions and which has at most $2p^2 n$ pairs of intersecting pairs. We obtain a subfamily \mathcal{F}' of \mathcal{F} containing no intersecting pairs by deleting at most $2p^2 n$ pairs from \mathcal{F} . It is easy to verify that if p is suitably small, \mathcal{F}' satisfies the requirements of the claim. \square

Let $\mathcal{F} = \{(x_j, y_j)\}_{j=1}^N$ be a family of pairs as in Claim 5.6 (so $N \leq pn$). We use the following simple technical claim, to avoid divisibility issues.

Claim 5.7. *Let β be suitably small and n suitably large. Then for some $1 \leq l \leq k$, there are at least βn^{4l-2} paths of length $4l - 1$ between each pair of vertices in F .*

Proof. Since F is (α, k) strongly robust, there exists $l \leq k$ such that between every $u, v \in V(F)$ there are at least αn^{l-2} paths in F of length $l - 1$. We conclude that for every $u, v \in V(F)$ there are at least $\alpha^{10} n^{4l-2}$ walks of length $4l - 1$ between u and v . Indeed, given $u, v \in V(F)$, there are at least $(\alpha n)^6$ ways to pick edges $e_1, e_2, e_3 \in E(F)$, since we may pick one end of each edge in at least $|F| \geq \alpha n$ ways, and then there are at least αn ways to pick a neighbour. Denote $e_i = (a_i, b_i)$. There are $(\alpha n^{l-2})^4$ ways to pick paths P_1, P_2, P_3, P_4 in F of length $l - 1$ with ends u and a_1, b_1 and a_2, b_2 and a_3, b_3 and v respectively. It follows that there are at least $\alpha^{10} n^{4l-2}$ walks in F between u and v . At most $O(n^{4l-3})$ of them are not paths, so for large enough n , there are at least $\frac{\alpha^{10}}{2} n^{4l-2}$ walks of length $4l - 2$ between u and v . \square

We shall build vertex-disjoint paths Q_j of length $8l^2 - 4l + 1$ one by one for $j = 1, \dots, N$ as follows. Suppose that Q_1, \dots, Q_{j-1} are already defined. We would like to pick paths as follows.

$P = (u_1, \dots, u_{4l})$ a path with ends x_j, y_j

P_i a path of length $4l - 1$ with ends u_i, u_{i+3} , for $i = 1, 3, \dots, 4l - 5$.

P_{4l-3} a path of length $4l - 1$ with ends u_{4l-3}, u_{4l-1} .

It is easy to see, by the choice of l according to Claim 5.7, that if p is small enough, we may pick such paths to be vertex-disjoint of all previously defined paths and to have pairwise disjoint interiors.

$$Q_j = (u_2 u_1 P_1 u_4 u_3 P_3 u_6 \dots u_{4l-2} u_{4l-3} P_{4l-3} u_{4l-1} u_{4l}).$$

Suppose that $w \in V(G)$ is a neighbour of $x_j = u_1$ and $y_j = u_{2l}$. The following path is a path in F with vertex set $V(Q_j) \cup \{w\}$ and same ends as Q_j (this path is illustrated in Figure 5 together with Q_j).

$$(u_2 u_3 P_3 u_6 u_7 P_7 \dots u_{4l-5} P_{4l-5} u_{4l-2} u_{4l-1} P_{4l-3} u_{4l-3} u_{4l-4} P_{4l-7} \dots u_5 u_4 P_1 u_1 w u_{4l}).$$

Finally, we let Q be a path which contains Q_1, \dots, Q_N by connecting the ends of the Q_j 's with paths of length $4l - 1$. Denote $\rho = 8pk^2$, and note that we may pick p small enough such that $\rho^2 \leq \frac{1}{16} p \alpha^2$. The length of Q is at most ρn and for every vertex $z \in V(F) \setminus V(Q)$, there are at least $\rho^2 n$ values of $j \in [N]$ such that $x_j, y_j \in N(z)$. We show that Q has the desired absorbing property. Let W be a set of at most $\rho^2 n$ vertices in $V(F) \setminus V(Q)$ and denote $W = \{w_1, \dots, w_M\}$. We may pick distinct $j_1, \dots, j_M \in [N]$ such that $x_{j_i}, y_{j_i} \in N(w_i)$. Recall that for each $i \in [M]$, there is a path Q'_{j_i} in F on vertex set $V(Q_{j_i}) \cup \{w_i\}$ with the same ends as Q_{j_i} . By replacing the occurrence of Q_{j_i} by Q'_{j_i} in the path Q , we obtain a path on vertex set $V(Q) \cup W$ with the same ends as Q .

We now turn to the proof of the second part of Lemma 5.5. Let F be an (α, k) weakly robust component with bipartition $\{X, Y\}$. The proof will use similar ideas, with some variations which take into account the bipartition of F . A similar argument as in Claim 5.6 implies that for small enough p we may find a family \mathcal{F} of disjoint quadruples of vertices of F with the following properties.

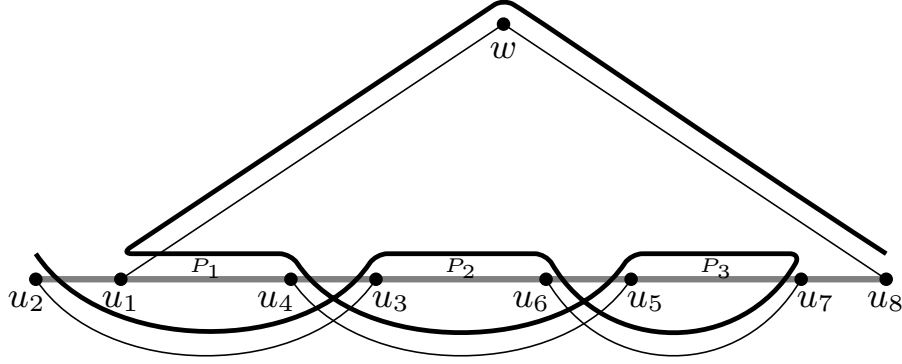


Figure 5: An illustration of the absorbing structure for $l = 2$. The path Q_j is represented by the straight line between u_2 and u_8 which is marked in grey, and the path absorbing w is represented by the bold black path.

- $|\mathcal{F}| \leq pn$
- For every $x \in X, y \in Y$ the number of quadruples (a, b, c, d) such that $a, c \in N(x)$ and $b, d \in N(y)$ is at least $\frac{1}{16}p\alpha^4 n$.

Denote $\mathcal{F} = \{(a^j, b^j, c^j, d^j)\}_{j=1}^N$. For $j = 1, \dots, N$ we pick a path Q_j as follows. As before, there exists $1 \leq l \leq k$ such that $|\text{con}_{F, 4l-2}(x, y)| \geq \beta n^{4l-2}$ for every $x \in X, y \in Y$. Assuming that Q^1, \dots, Q^{j-1} were already chosen to be paths of length at most $10l^2$, we may pick paths

$$\begin{aligned} & (a_1 b_1 \dots a_{2l} b_{2l}) \text{ a path with ends } a_1 = a^j, b_{2l} = b^j \\ & (c_1 d_1 \dots c_{2l} d_{2l}) \text{ a path with ends } c_1 = c^j, d_{2l} = d^j \end{aligned}$$

which are vertex-disjoint of each other and of previously defined paths. We may further pick paths of length at most k

$$\begin{aligned} P_i & \text{ a path with ends } \begin{cases} a_1, c_{2l} & \text{if } i = 1 \\ a_i, b_{i+1} & \text{if } 2 \leq i \leq 2l - 2 \\ a_{2l-1}, a_{2l} & \text{if } i = 2l - 1 \end{cases} \\ Q_i & \text{ a path with ends } \begin{cases} d_2, d_1 & \text{if } i = 2 \\ d_i, c_{i-1} & \text{if } 3 \leq i \leq 2l \end{cases} \\ R & \text{ a path with ends } c_1, b_2. \end{aligned}$$

which are pairwise vertex-disjoint and are disjoint of previously defined paths. Let

$$Q^j = (b_1 a_1 P_1 c_{2l} d_{2l} Q_{2l} c_{2l-1} \dots d_3 Q_3 c_2 d_2 Q_2 d_1 c_1 R b_2 a_2 P_2 b_3 \dots a_{2l-2} P_{2l-2} b_{2l-1} a_{2l-1} P_{2l-1} a_{2l} b_{2l}).$$

Suppose that x, y satisfy $a^j, c^j \in N(x)$ and $b^j, d^j \in N(y)$. Then the following path is a path in F

with vertex set $V(Q^j) \cup \{x, y\}$ with the same ends as Q^j (See figure 6).

$$(b_1 a_2 P_2 b_3 a_4 P_4 b_5 \dots a_{2l-2} P_{2l-2} b_{2l-1} a_{2l} P_{2l-1} a_{2l-1} b_{2l-2} P_{2l-3} a_{2l-3} \dots b_4 P_3 a_3 b_2 R c_1 x \\ a_1 P_1 c_{2l} d_{2l-1} Q_{2l-1} c_{2l-2} \dots d_3 Q_3 c_2 d_1 Q_2 d_2 c_3 Q_4 d_4 \dots c_{2l-1} Q_{2l} d_{2l} y b_{2l}).$$

We may proceed as in the previous part to complete the proof of Lemma 5.5. □

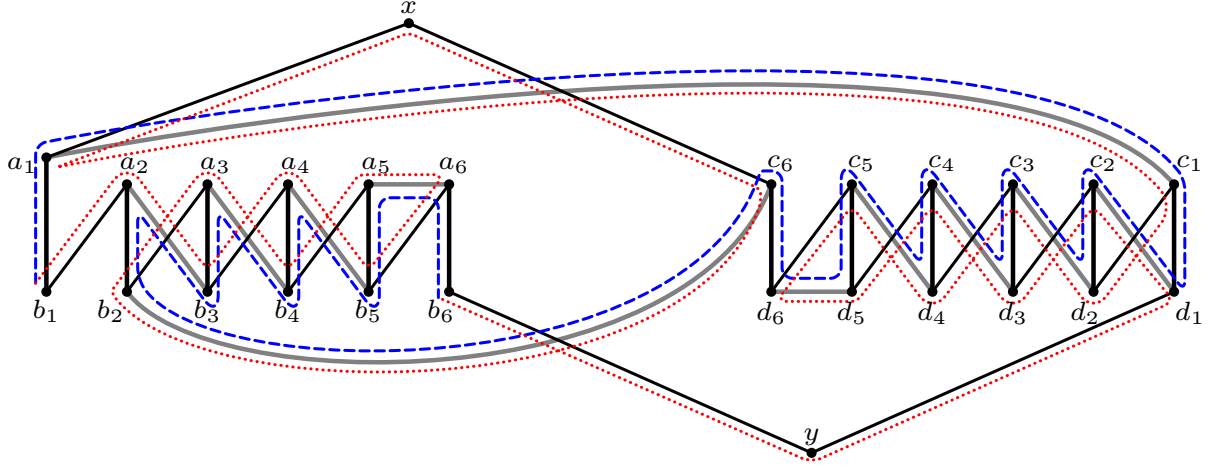


Figure 6: An illustration of the absorbing structure for $l = 3$. The black lines represent edges, whereas the grey ones represent paths. The dashed blue line represents the path Q^j and the dotted red one represents the path obtained from Q^j by absorbing w .

The proof of Lemma 5.5 concludes our introduction of the notion of robust subgraphs and their properties. In order to make use of the properties we established, we shall use Lemmas 6.3, 6.7 and 6.8 from the next section, Section 6. These lemmas establish the connection between connected components of the reduced graph (obtained by an application of the Regularity Lemma) and robust subgraphs.

6 The Regularity Lemma

In our proof, we shall use Szemerédi's Regularity Lemma extensively. Before stating the version we use, we introduce some notation. Let U, W be disjoint subsets of vertices of a graph G . The density $d(U, W)$ of edges between U and W is defined to be

$$d(U, W) = \frac{e(U, W)}{|U||W|},$$

where $e(U, W)$ is the number of edges between U and W . A bipartite graph with bipartition U, W is said to be ε -regular if for every $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \geq \varepsilon|U|$ and $|W'| \geq \varepsilon|W|$,

the density $d(U', W')$ satisfies $|d(U', W') - d(U, W)| \leq \varepsilon$.

We use a variant of the so-called degree form of the Regularity Lemma (see [19]), which is applicable to 2-coloured graphs. Furthermore, it will be useful for our purpose to start with a cover of the vertices of a graph and require that the partition obtained by the lemma is a refinement of the initial cover.

Lemma 6.1. *For every $\varepsilon > 0$ and integer l there exists $M = M(l, \varepsilon)$ such that the following holds. Let G be a 2-coloured graph on n vertices, \mathcal{C} a cover of $V(G)$ with at most l parts and $d > 0$. Then there exists a partition $\{V_0, \dots, V_m\}$ of $V(G)$ and a subgraph G' of G with vertex set $V(G) \setminus V_0$, such that the following conditions hold.*

(R1) $m \leq M$.

(R2) Every V_i ($i \in [m]$) is contained in one of the parts of \mathcal{C} .

(R3) $|V_0| \leq \varepsilon n$ and $|V_1| = \dots = |V_m| \leq \lceil \varepsilon n \rceil$.

(R4) $\deg_{G'}(v) \geq \deg_G(v) - (2d + \varepsilon)n$ for every $v \in V_1 \cup \dots \cup V_m$.

(R5) $e(G'[V_i]) = 0$ for $i \in [m]$.

(R6) All pairs (V_i, V_j) are ε -regular in both colours in G' , with density in each colour either 0 or at least d .

It is often useful to work with the reduced graph, obtained from the partition given by the Regularity Lemma as follows. Given a 2-coloured graph G , and parameters ε, d, l , we define the (ε, d) -reduced graph Γ as follows. Let $\{V_0, \dots, V_m\}$ be the partition obtained by an application of Lemma 6.1, and let G' be the given subgraph of G . We take $V(\Gamma) = \{V_1, \dots, V_m\}$. A pair $V_i V_j$ is a t -coloured edge in Γ if it has density at least d in colour t in G' . Note that an edge of Γ can have more than one colour.

The following observation shows why it is useful to work with the degree form of the Regularity Lemma.

Observation 6.2. *Let G be a 2-coloured graph on n vertices with $\delta(G) \geq cn$ and let Γ be the (ε, d) -reduced graph obtained by applying Lemma 6.1. Then $\delta(\Gamma) \geq (c - 2d - \varepsilon)m$, where $m = |\Gamma|$.*

The rest of this section is divided into two parts. In the first part, Subsection 6.1, we establish the connection between robust subgraph and connected components of a reduced graph. In the second part, Subsection 6.2, we describe the connection between connected matchings in a reduced graph and cycles in the original graph.

6.1 From connected components of the reduced graph to robust subgraphs

One of our main tools in the proof of Theorem 1.1 is the following lemma. It gives us the means to obtain a robust subgraph from a connected subgraph of the reduced graph.

Lemma 6.3. *Let $\alpha, \frac{1}{k}, \frac{1}{n} \ll \varepsilon, \frac{1}{l} \ll 1$ and $d \geq 4\varepsilon$. Let G be a graph on n vertices and let Γ be the (ε, d) -reduced graph obtained by an application of Lemma 6.1. Suppose that Φ is a connected subgraph of Γ . Then there exists a subgraph F of G with the following properties.*

1. Let U be the set of vertices contained in clusters of $V(\Phi)$. Then $V(F) \subseteq U$ and $|U \cap V(F)| \geq (1 - \varepsilon)|U|$.
2. F is (α, k) -robust. If Φ is bipartite, F is weakly robust, otherwise it is strongly-robust.

In order to find the required robust component F , we consider the clusters represented by $V(\Phi)$, and for each of the clusters we remove vertices with low degree. The regularity of pairs of clusters which are connected by an edge, together with the choice of d , implies that the number of low degree vertices in each cluster is small. We show that the subgraph induced by the remaining vertices has the required expansion properties, using the regularity of the pairs of clusters.

Proof of Lemma 6.3. Denote $I = \{i \in [m] : V_i \in V(\Gamma)\}$, so $U = \bigcup_{i \in I} V_i$. For every $i \in I$, denote by I_i the set of indices $j \in I$ such that $V_i V_j \in E(\Phi)$ and let $N_i = \bigcup_{j \in I_i} V_j$. Let G' be the subgraph of G given by Lemma 6.1). Let

$$W_i = \{v \in V_i : \deg_H(v, N_i) \leq 3\varepsilon|N_i|\}.$$

Claim 6.4. $|W_i| \leq \varepsilon n$ for every $i \in I$.

Proof. Suppose otherwise. Recall that for every $j \in I_i$, (V_i, V_j) is an ε -regular pair in G' with density at least 4ε . It follows by the definition of a regular pair that $d_{G'}(W_i, V_j) \geq d_{G'}(V_i, V_j) - \varepsilon \geq 3\varepsilon$. Hence, $e_{G'}(W_i, V_j) \geq 3\varepsilon|W_i||V_j|$ for $j \in I_i$ and $e_{G'}(W_i, N_i) \geq 3\varepsilon|W_i||N_i|$. It follows that there exists a vertex in W_i which is incident to at least $3\varepsilon|N_i|$ edges into N_i , a contradiction to the choice of W_i . \square

Define $W = \bigcup_{i \in I} W_i$ and $F = G_B[U \setminus W]$. Note that F satisfies Property (1) in Lemma 6.3. It remains to show that Property (2) holds. We suppose that Φ is non-bipartite, the proof for the bipartite case follows similarly. We use the following simple claim.

Claim 6.5. *Let G be a connected non-bipartite graph on n vertices. Then there exists $k \leq 3n$ such that between every two vertices of G there is a walk of length k .*

Proof. We first show that between every two vertices of G there is a walk of odd length not exceeding $3n$. Indeed, let $x, y \in V(G)$. Let C be an odd cycle, and pick some $z \in C$. Pick some path from x to z and a path from z to y . Combining the two paths, we obtain a walk from x to y of length at most $2n$. If this walk has even length, we add the cycle C to it. In any case we obtain an odd walk of length at most $3n$. Let k be the length of the longest of these walks. By possibly adding 2-cycles to the given walks, we obtain walks of length k between each pair of vertices. \square

Fix some $k \leq 3n$ as in the previous claim. In the following claim, we show that the existence of a walk of length k in Φ implies the existence of many paths of the same length in F .

Claim 6.6. *Let $k \leq 3m$. There exists $\beta = \beta(l, \varepsilon)$ such that the following holds. Suppose that V_{i_1}, \dots, V_{i_k} is a walk in Φ . Let U_1 and U_k be subsets of $V_{i_1} \cap V(F)$ and $V_{i_k} \cap V(F)$ respectively of size at least $2\varepsilon|V_1|$. Then there exists at least βn^{k+1} paths of length k in F between U_1 and U_k .*

Proof. Let $X_j = \{v \in V_{i_j} : \deg_H(v, V_{i_{j+1}}) \leq 3\varepsilon|V_1|\}$ for $j \in [k-1]$. As before it is easy to show that $|X_j| \leq \varepsilon|V_1|$. Let

$$Y_j = \begin{cases} U_1 \setminus X_1 & j = 1 \\ V_{i_j} \setminus (X_j \cup W_{i_j}) & j \in [2, k-1] \end{cases}$$

Recall that $|W_j| \leq \varepsilon|V_1|$ for every j . It follows that for $j \in [k-1]$ and $v \in Y_j$, we have $\deg_H(v, Y_{j+1}) \geq \varepsilon|V_1|$. Thus there are at least $(\varepsilon|V_1|)^{k-2}$ walks of length $k-3$ between Y_1 and Y_{k-2} . Fix any such walk $(v = v_1, \dots, v_{k-2})$. Denote by A the neighbourhood of v_{k-2} in Y_{k-1} . Then $|A| \geq \varepsilon|V_1|$ and recall that $|U_k| \geq \varepsilon|V_1|$. Since $(V_{i_{k-1}}, V_{i_k})$ is an ε -regular pair in H with density at least 4ε , we have $e(A, U_k) \geq 3\varepsilon|A||U_k| \geq 3\varepsilon^3|V_1|^2$. Each such edge completes the above walk into a distinct walk in H between v_1 and U_k of length $k-1$.

We conclude that the total number of walks of length $k-1$ between U_1 and U_k is at least $\varepsilon^{k+1}|V_1|^k \geq \varepsilon^{k+1}(\frac{1-\varepsilon}{M})^k n^k$, where M is as in Lemma 6.1. Note that the number of such walks which are not paths (namely, a vertex appears more than once) is $O(n^{k-1})$. It follows that indeed, there are at least βn^k paths between U_1 and U_k of length $k-1$, where β is a suitable constant. \square

Let $x, y \in V(F)$ and suppose that $x \in V_i$ and $y \in V_j$. Note that the definition of F implies the existence of $t, s \in I$ such that $(V_i, V_t), (V_j, V_s) \in E(\Phi)$ and x and y have at least $2\varepsilon|V_1|$ neighbours in H in V_t and V_s respectively. It follows from Claim 6.6 that there are at least βn^k paths from the neighbourhood of x in V_t to the neighbourhood of y in V_s . This shows that $|con_{F,k}(x, y)| \geq \frac{\beta}{2} n^k$ for every $x, y \in V(F)$, implying that F is $(\beta/2, m)$ strongly robust. \square

In fact, we need a stronger version of Lemma 6.3. In our application, since we deal with 2-coloured graph, we will typically have two collections of connected subgraphs of Γ , one for each colour, and it would be useful to obtain collections of robust components which preserve containment. For example, if in the reduced graph we have blue components Φ_1, Φ_2 and red components Φ_3, Φ_4 satisfying $V(\Phi_1) \cup V(\Phi_2) = V(\Phi_3) \cup V(\Phi_4)$, we would like the corresponding robust subgraphs F_1, F_2, F_3, F_4 to satisfy the corresponding equality, namely $V(F_1) \cup V(F_2) = V(F_3) \cup V(F_4)$. This is achieved by the following lemma. We remark that the proof is very similar to the previous one, so we omit it.

Lemma 6.7. *Let $\alpha, \frac{1}{k}, \frac{1}{n} \ll \varepsilon, \frac{1}{l} \ll 1$ and $d \geq 6\varepsilon$. Let G be a graph on n vertices with a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Let Γ be the (ε, d) -reduced graph obtained by an application of Lemma 6.1 and let $\{V_0, \dots, V_m\}$ be the corresponding partition of $V(G)$. Let \mathcal{P}_B and \mathcal{P}_R be collections of disjoint connected subgraphs of Γ_B and Γ_R respectively. Then there exist subsets $U_i \subseteq V_i$ for $i \in [m]$ satisfying the following properties.*

(i) $|U_i| \geq (1 - 2\varepsilon)|V_i|$ for $i \in [m]$.

(ii) Let $\Phi \in \mathcal{P}_t$, where $t \in \{B, R\}$ and denote $I = \{i \in [m] : V_i \in V(\Phi)\}$. Then the graph $F = G_t[\bigcup_{i \in I} U_i]$ is (α, k) -robust. If Φ is bipartite, F is weakly robust, otherwise it is strongly-robust.

In our proof we shall find robust components and then apply the Regularity Lemma. Therefore we need the following result, stating that given a robust component F in G , the corresponding subgraph of F in the reduced graph Γ is connected.

Lemma 6.8. *Let $\varepsilon, \frac{1}{n} \ll \alpha, \frac{1}{k}, \frac{1}{l} \ll 1$ and let G be a graph on n vertices with a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that F is an (α, k) -robust component of G_t , where $t \in \{B, R\}$ and let \mathcal{C} be a cover of $V(G)$ with at most l parts refining $\{V(F), V(G) \setminus V(F)\}$. Let Γ be the (ε, d) -reduced graph obtained by an application of Lemma 6.1. Then the t -coloured subgraph Φ of Γ spanned by the clusters contained in $V(F)$ is connected.*

Proof. Let G' be the corresponding subgraph of G obtained by applying Lemma 6.1. Let U be the set of vertices of F which belong to sets V_i contained in $V(F)$. Denote $F' = G'_t[U]$. Recall that by Lemma 6.1, we have that $|V(F) \setminus U| \leq \varepsilon n$ and $\deg_{G'}(v) \geq \deg_G(v) - 9\varepsilon n$. In particular $\deg_{F'}(v) \geq \deg_F(v) - 9\varepsilon n$ for every $v \in V(F')$. Hence F' is obtained by removing at most εn vertices of F and then removing a subgraph with maximum degree at most $9\varepsilon n$. By Lemmas 5.2, 5.3, we have that F' is $(\alpha/4, k)$ -robust in G' . In particular, F' is connected and it follows that Φ (the t -coloured subgraph of Γ spanned by clusters contained in $V(F)$) is connected. \square

6.2 From connected matchings to long cycles

We shall use the technique of converting connected matchings in the reduced graph into cycles in the original graph. This was introduced by Łuczak [21], and since then has become fairly standard (see [3], [8] and [15], [16], [17] and [22]).

For the sake of completeness, we prove the following result, stating that given a connected matching in the reduced graph, there exists a cycle in the original graph through most of the vertices in the clusters and few additional vertices.

Lemma 6.9. *Let $\varepsilon > 0$ and $d \geq 3\varepsilon$ and let n be suitably large. Let G be a graph on n vertices and let Γ be the (ε, d) -reduced graph obtained by an application of Lemma 6.1. Suppose that \mathcal{M} is a connected matching in Γ and denote by U the set of vertices spanned by the clusters of \mathcal{M} . Then G contains a cycle covering at least $(1 - 6\varepsilon)|U|$ of the vertices of U .*

Let us first sketch the proof. Using the fact that \mathcal{M} is connected we can connect the matching edges by paths, following a cyclic ordering of the edges in \mathcal{M} . We replace these paths by short vertex-disjoint paths in the original graph G between the cluster pairs associated to the edges of \mathcal{M} . These connecting paths will be parts of the final cycle. To define the rest of the cycle, remove the internal vertices of these connecting paths, and in each cluster pair find a long path to close the connecting pairs to a cycle. There are various ways to find these long cycles, e.g. using the rotation-extension technique of Pósa [24] or as a very special case of the Blow-up Lemma [18]. We shall use a much simpler result to obtain these cycles.

Proof of Lemma 6.9. Denote by $\{V_0, \dots, V_m\}$ the corresponding partition of $V(G)$ and $\mathcal{M} = (V_{i_1}, V_{j_1}), \dots, (V_{i_m}, V_{j_m})$. It is easy to see that we may find paths P_1, \dots, P_m with the following properties.

- The paths P_l are vertex disjoint.

- Each path P_l contains at most one vertex from each cluster V_i .
- Denote the ends of P_l by x_l, y_l . Then

$$\begin{aligned} |N(x_l) \cap V_{j_l}| &\geq 2\varepsilon|V_1| \\ |N(y_l) \cap V_{i_{l+1}}| &\geq 2\varepsilon|V_1|. \end{aligned}$$

Given such paths P_1, \dots, P_m , denote $U = V(P_1) \cup \dots \cup V(P_m)$. Pick sets

$$\begin{aligned} A_l &\subseteq (N(x_l) \cap V_{j_l}) \setminus U \\ B_l &\subseteq (N(y_l) \cap V_{i_{l+1}}) \setminus U \end{aligned}$$

of size $\varepsilon|V_1|$. Note that this is indeed possible because by the choice of x_l, y_l and since for large enough n , $|U| \leq \varepsilon|V_1|$. Pick sets

$$\begin{aligned} C_l &\subseteq N(x_l) \setminus (U \cup A_l) \\ D_l &\subseteq N(y_l) \setminus (U \cup B_l) \end{aligned}$$

of size $(1 - 2\varepsilon)|V_1|$ each. We shall show that for each l , $G[C_l, D_l]$ contains a path Q_l missing at most $2\varepsilon|V_1|$ vertices from each set C_l, D_l . Assuming that we may find such paths, it is now easy to construct the required cycle. By the definition of the reduced graph, the density of edges between subsets of V_{i_l} and V_{j_l} of size at least $\varepsilon|V_1|$ is positive. Thus there is an edge between A_l and the last $2\varepsilon|V_1|$ vertices of Q_l as well as between B_l and the first $2\varepsilon|V_1|$ vertices of Q_{l+1} . Thus by losing at most $4\varepsilon|V_1|$ vertices from each path Q_l we may use the paths P_l to obtain a cycle C which misses at most $6\varepsilon|V_1|$ vertices from each cluster in $V(\mathcal{M})$.

It remains to show that such paths Q_l may be found. We use the following simple but useful claim which was proved independently by Pokrovskiy [23] and Dudek and Prałat [9]. For the sake of completeness, we include the proof here.

Proposition 6.10. *For every graph G there exist two disjoint sets of vertices U, W of equal size such that G has no edges between U and W and the graph $G \setminus (U \cup W)$ has a Hamilton path.*

Proof. In order to find sets with the desired properties, we apply the following algorithm, maintaining a partition of $V(G)$ into sets U, W and a path P . Start with $U = V(G)$, $W = \emptyset$ and P an empty path. At every stage in the algorithm, do the following. If $|U| \leq |W|$, stop. Otherwise, if P is empty, move a vertex from U to P (note that $U \neq \emptyset$). If P is non-empty, let v be its endpoint. If v has a neighbour u in U , put u in P , otherwise move v to W . Note that at any given point in the algorithm there are no edges between U and W . Furthermore, the value $|U| - |W|$ is positive at the beginning of the algorithm and decreases by one at every stage, thus at some point the algorithm will stop and will produce sets U, W with the required properties. \square

We need the following corollary of Proposition 6.10.

Corollary 6.11. *Let G be a balanced bipartite graph on n vertices with bipartition V_1, V_2 , which has no path of length k . Then there exist $X_i \subseteq V_i$ such that $|X_1| = |X_2| \geq (n - k)/4$ and G has no edges between X_1 and X_2 .*

Proof. Let U, W be as in Proposition 6.10 and let P be a Hamilton path in $G \setminus (U \cup W)$. Note that P must alternate between V_1 and V_2 , thus $|V(P) \cap V_1| = |V(P) \cap V_2|$ (by the assumptions, the number of vertices in P is even). Denote $U_i = U \cap V_i$ and $W_i = W \cap V_i$ for $i = 1, 2$, and assume that $|U_1| \geq |U_2|$. It follows that $|U_1| + |W_1| = |U_2| + |W_2|$. Thus, using the fact that $|U| = |W|$, we have $|U_1| = |W_2| \geq |U|/2 \geq (n - k)/4$. Set $X_1 = U_1$ and $X_2 = W_2$. \square

The graph $G[C_l, D_l]$ is a balanced bipartite graph, satisfying that for every choice of sets $C' \subseteq C_l, D' \subseteq D_l$ of size at least $\varepsilon|V_1|$ each, we have $e(G[C', D']) > 0$. It follows from Corollary 6.11 that $G[C_l, D_l]$ contains a path missing at most $2\varepsilon|V_1|$ vertices from each side. As explained previously, Lemma 6.9 follows. \square

In fact, we shall need a slight generalisation of Lemma 6.9. The usual setting in which we apply the described technique is as follows. We consider the reduced graph obtained by applying the Regularity Lemma. In the reduced graph, we find a perfect matching consisting of a connected blue matching and a connected red matching, and we use the described technique to find two disjoint cycles, one blue and one red, which together cover almost all of the vertices.

To obtain disjoint cycles, we set aside small sets of each cluster of the red component which will be used to connect the cluster pairs of the red connected matching. We then find a blue cycle as above covering most of the clusters of the blue matching, and then we find a red cycle similarly, using the sets we set aside.

We shall not write the exact statement of the modified version of Lemma 6.9, but let us remark that the constants change slightly. We need $d \geq 6\varepsilon$ and are able to cover $(1 - 9\varepsilon)n$ of the vertices of each cluster in the matchings.

Finally, we point out that we often first find “absorbing paths” in the original graph and then apply the Regularity Lemma to the remaining graph. When building the cycles obtained by the connected matchings, we would like them to contain these predefined paths. This is obtained by the same method, enabling us to find a path (rather than a cycle) between the neighbourhoods of the two ends of the absorbing path.

This concludes the introduction of the tools we shall need for our proof of Theorem 1.1. To complete the preliminary material needed for our proof, we list several extremal results in the next section.

7 Extremal results

In this section we list a number of extremal results we shall use in our proofs. They concern mainly with the existence of matchings, paths and cycles in graphs with certain structural conditions.

The following is Chvátal’s theorem [7] giving sufficient conditions on the degree sequence of a graph for containing a Hamilton cycle.

Theorem 7.1. *Let G be a graph on $n \geq 3$ vertices and let $d_1 \leq \dots \leq d_n$ be the degree sequence of G . Suppose that $d_i \geq i + 1$ or $d_{n-i} \geq n - i$ for every $i \leq n/2$. Then G contains a Hamilton cycle.*

In some cases it is easier to use the following version of Chvátal's result for bipartite graphs.

Corollary 7.2. *Let G be a balanced bipartite graph on $2n$ vertices with bipartition $\{X, Y\}$. Let $x_1 \leq \dots \leq x_n$ be the degree sequence of X and let $y_1 \leq \dots \leq y_n$ be the degree sequence of Y . Suppose that $x_i \geq i + 1$ or $y_{n-i} \geq n - i + 1$ for every $i \in [n]$. Then G contains a Hamilton cycle.*

Proof. Consider the graph G' obtained from G by adding all edges with both ends in X . By Theorem 7.1, G' contains a Hamilton cycle C . As $|X| = |Y|$, the cycle C contains no edges with both ends in X , i.e. C is a Hamilton cycle in G . \square

The following is a simple result by Erdős and Gallai [10], giving an upper bound on the number of edges in a graph with no path of a given length.

Theorem 7.3. *Let G be a graph on n vertices with no paths of length at least $l + 1$, then $e(G) \leq nl/2$.*

A graph on n vertices is called *pancyclic* if for every $l \leq n$, G contains a cycle on l vertices. The following result by Bondy [6] is a generalisation of Dirac's Theorem, asserting that graphs with large enough minimum degree are pancyclic.

Theorem 7.4. *Let G be a graph on n vertices with $\delta(G) > n/2$. Then G is pancyclic.*

In the following two subsections, we use the following well known theorem of Tutte, giving a necessary and sufficient condition for having a perfect matching.

Theorem 7.5. *Let G be a graph on an even number of vertices. Then G has a perfect matching if and only if for every set of vertices U , the number of odd components of $G \setminus U$ is at most $|U|$.*

7.1 Matchings in tripartite graphs

In section 14 we shall analyse conditions for certain tripartite graph to have a perfect matching. Here we describe the extremal results we shall need for the analysis. We use a stability version of the following lemma of DeBiasio and Nelsen [8]. For the sake of completeness, we prove it here.

Lemma 7.6. *Let n be even, and let G be a tripartite graph on n vertices with tripartition $\{X_1, X_2, X_3\}$. Suppose that $|X_i| \leq n/2$ and $\deg(x) > 3n/4 - |X_i|$ for every $x \in X_i$, $i \in [3]$. Then G has a perfect matching.*

Proof. Suppose to the contrary that G has no perfect matching. Without loss of generality, we assume that G is a maximal counter example. It is easy to check that the graph obtained from the complete tripartite graph with partition $\{X_1, X_2, X_3\}$ by removing any edge has a perfect matching. Thus we may assume that G is not complete as a tripartite graph. Without loss of generality, suppose that $v_1v_2 \notin E(G)$ where $v_1 \in X_1, v_2 \in X_2$. It follows from the maximality of G that the addition of v_1v_2 to G completes a perfect matching M in G . Clearly, $v_1v_2 \in M$.

M contains no edge $u_1u_2 \neq v_1v_2$ where $u_i \in N(v_i)$ for $i = 1, 2$, otherwise there exists a perfect matching in G (replace v_1v_2, u_1u_2 by v_1u_1, v_2u_2). Denote by M_1 the edges of M which have ends in

X_1 and X_2 , and let $M_2 = M \setminus M_1$. It follows that for every $e \in M_1 \setminus \{v_1 v_2\}$, $\deg(v_1, e) + \deg(v_2, e) \leq 1$ and for every $e \in M_2$, $\deg(v_1, e) + \deg(v_2, e) \leq 2$.

Thus

$$\begin{aligned} \deg(v_1) + \deg(v_2) &= \sum_{e \in M} (\deg(v_1, e) + \deg(v_2, e)) \leq \\ &(|M_1| - 1) + 2|M_2| = |M| - 1 + |M_2| = n/2 - 1 + |X_3|. \end{aligned}$$

For the last equality, we used the fact that $|M_2| = |X_3|$. By the degree condition on G , we have $\deg(v_1) + \deg(v_2) > 3n/2 - |X_1| - |X_2| = n/2 + |X_3|$, a contradiction to the above inequality. \square

The following lemma is a stability version of Lemma 7.6. We prove it by applying Tutte's theorem, Theorem 7.5.

Lemma 7.7. *Let $\frac{1}{n} \ll \varepsilon \ll 1$, where n is even. Let G be a graph on n vertices with tripartition $\{X_1, X_2, X_3\}$ such that $|X_i| \leq (1/2 - 4\varepsilon)n$ and $\deg(x, V(G) \setminus X_i) \geq (3/4 - \varepsilon)n - |X_i|$ for $i \in [3]$ and $x \in X_i$. Then one of the following holds.*

- G has a perfect matching.
- There exists an independent set Y in G such that $Y \subseteq X_i \cup X_j$ and $|Y \cap X_i|, |Y \cap X_j| \geq (1/4 - 5\varepsilon)n$ for some $1 \leq i < j \leq 3$.

Proof. We assume that G has no perfect matching. By Tutte's theorem, Theorem 7.5, there exists a subset $S \subseteq V(G)$ such that the number of odd components of $G \setminus S$ is larger than $|S|$.

Denote $G' = G \setminus S$. We show first that $\delta(G') \leq \varepsilon n$. Suppose not. Then the number of components of G' is at most $1/\varepsilon$, thus $|S| \leq 1/\varepsilon$. We show that G' is connected, contradicting the choice of S . Given $u, v \in X_i$, they are non-adjacent to at most $(1/4 + \varepsilon)n$ vertices in $V(G) \setminus X_i$. But $|V(G) \setminus X_i| \geq (1/2 + 4\varepsilon)n$, hence u, v have at least $2\varepsilon n \geq 1/\varepsilon$ common neighbours. It follows that indeed, G' is connected.

By the assumptions on G , $\delta(G) \geq (3/4 - \varepsilon)n - \max\{|X_1|, |X_2|, |X_3|\} \geq (1/4 + 3\varepsilon)n$. Since $\delta(G') \leq \varepsilon n$, we have $|S| \geq (1/4 + 2\varepsilon)n$.

Note that $|S| \leq n/2$, because the number of odd component of G' is at most $n - |S|$. Denote $X'_i = X_i \setminus S$. We show that $|X'_i| \leq (1/4 + 2\varepsilon)n$ for $i \in [3]$. Indeed, suppose that $|X'_1| \geq (1/4 + 2\varepsilon)n$. Then for every vertex $u \in X'_2 \cup X'_3$ we have $\deg_{G'}(u) \geq \varepsilon n$. In particular, u is in a component of G' order at least εn . Furthermore, every non isolated vertex of X'_1 is adjacent to some vertex in $X'_2 \cup X'_3$, and thus is in a component of size at least εn . Since $X'_2 \cup X'_3$ is non empty (e.g. because $|S| \leq n/2$), it follows that G' has at most $(1/4 + \varepsilon)n$ isolated vertices, and the rest of the vertices are in components of order at least εn . Hence the number of odd components of G' is at most $(1/4 + \varepsilon)n + 1/\varepsilon \leq |S|$, a contradiction.

Denote $S_i = X_i \cap S$, $i \in [3]$. Consider the three quantities $|X_1| + |S_2| + |S_3|, |X_2| + |S_1| + |S_3|, |X_3| + |S_1| + |S_2|$. Their sum is $n + 2|S| \leq 2n$. Without loss of generality, it follows that $|X_1| + |S_2| + |S_3| \leq 2n/3$. Thus for every $u \in X'_1$,

$$\deg_{G'}(u) \geq (3/4 - \varepsilon)n - |X_1| - |S_2| - |S_3| \geq (1/12 - \varepsilon)n.$$

If in addition $|X'_1| + |X'_2| \geq (1/4 + 2\varepsilon)n$, every vertex in X'_3 is in a component of order at least εn . But the number of isolate vertices in X'_2 is at most $(1/4 + \varepsilon)n$, so the number of odd components in G' is at most $(1/4 + \varepsilon)n + 1/\varepsilon \leq |S|$. We conclude that $|X'_1| + |X'_2| \leq (1/4 + 2\varepsilon)n$. Recall that $|X'_3| \leq (1/4 + 2\varepsilon)n$, implying that $|V(G')| \leq (1/2 + 4\varepsilon)n$, i.e. $|S| \geq (1/2 - 4\varepsilon)n$. If $X'_1 \neq \emptyset$, G' contains a component of order at least $(1/12 - \varepsilon)n$ in G' , so there are at most $(5/12 + 5\varepsilon)n + 1 \leq |S|$ components in G' , a contradiction.

The set Y obtained by picking one vertex from each component of G' is an independent set in $X'_2 \cup X'_3$ of size at least $|S|$. Let $Y_i = Y \cap X'_i$. Then G contains no $Y_2 - Y_3$. Recall that each vertex in Y_2 has at most $(1/4 + \varepsilon)n$ non neighbours in Y_3 , implying that $|Y_3| \leq (1/4 + \varepsilon)n$ and $|Y_2| \geq (1/4 - 5\varepsilon)n$. By symmetry, we also have $|Y_3| \geq (1/4 - 5\varepsilon)n$. \square

7.2 Matchings in bipartite graphs

The following lemma is a simple consequence of Hall's theorem for perfect matchings in bipartite graphs.

Lemma 7.8. *Let G be a balanced bipartite graph with bipartition $\{X_1, X_2\}$ on n vertices with $\delta(G) \geq (1/4 - \varepsilon)n$. Then one of the following conditions hold.*

- G contains a perfect matching.
- There are subsets $A_i \subseteq X_i$ of size $(1/4 - \varepsilon)n \leq |A_1|, |A_2| \leq (1/4 + \varepsilon)n$ such that $G[A_1, A_2]$ contains no edges.

Proof. Suppose that G contains no perfect matching. Then by Hall's Theorem there exists $A_1 \subseteq X_1$ such that $|N(A_1)| < |A_1|$. Denote $A_2 = X_2 \setminus N(A_1)$. Then $G[A_1, A_2]$ has no edges. Note that since $\delta(G) \geq (1/4 - \varepsilon)n$, we have $|A_1| > |N(A_1)| \geq (1/4 - \varepsilon)n$. By symmetry we have that $|A_2| \geq (1/4 - \varepsilon)n$. Thus $(1/4 - \varepsilon)n \leq |A_1| = |A_2| \leq (1/4 + \varepsilon)n$. \square

In Section 14, we shall also need the following stability result for graphs with a bipartition satisfying certain conditions. The proof is again an applications of Tutte's theorem, Theorem 7.5.

Lemma 7.9. *Let $\frac{1}{n} \ll \varepsilon \ll 1$, where n is even. Let G be a graph on n vertices, and suppose that $\{X_1, X_2\}$ is a partition of $V(G)$ such that $|X_i| \geq (1/2 - \varepsilon)n$ and $\deg(u, X_{3-i}) \geq (3/4 - \varepsilon)n - |X_i|$ for every $i \in [3]$, $u \in X_i$. Then one of the following conditions holds.*

- G has a perfect matching.
- G is not 2-connected.
- There exists an independent set $A_i \subseteq X_i$ (for some $i \in [2]$) of order at least $(1/4 - 4\varepsilon)n$ such that $|N(A_i)| \leq (1/4 + 3\varepsilon)n$.
- $|X_i| > |X_{3-i}|$ and X_i contains an independent set of size at least $(1/2 - \varepsilon)n$.

- *There exists an independent set A of size at least $(1/2 - 6\varepsilon)n$ such that $|A \cap X_i| \geq (1/4 - 9\varepsilon)n$.*

Proof. Suppose that G has no perfect matching. It follows from Tutte's theorem, Theorem 7.5, that there exists a set S such that the number of odd components in $G' = G \setminus S$ is larger than $|S|$.

Denote $S_i = S \cap X_i$ and $X'_i = X_i \setminus S$. If $\delta(G') \geq \varepsilon n$, the number of components of G' is at most $1/\varepsilon$, implying that $|S| \leq 1/\varepsilon$. In this case, the conditions on the graph $G[X_1, X_2]$ imply that either G' is connected, contradiction our assumption on S , or G' consists of two connected components, implying that $|S| \leq 1$, so G is not 2-connected.

We now assume that $\delta(G') \leq \varepsilon n$. It follows that $|S| \geq (1/4 - 2\varepsilon)n$. Suppose that $|X'_1| \geq (1/4 + 3\varepsilon)n$ and $X'_2 \neq \emptyset$. Denote by A_1 the set of isolated vertices in X'_1 . Then every vertex in $X'_2 \cup (X'_1 \setminus A_1)$ belongs to a component of G' of size at least εn . Furthermore, $|A_1| \leq (1/4 + 2\varepsilon)n$. It follows that the number of components is at most $|A_1| + 1/\varepsilon \leq (1/4 + 3\varepsilon)n$, implying that $|S| \leq (1/4 + 3\varepsilon)n$. But $|S| \geq (1/4 - 2\varepsilon)n$, so $|A_1| \geq |S| - 1/\varepsilon \geq (1/4 - 3\varepsilon)n$. Clearly, $N(A_1) \subseteq S$, and the third conditions of the lemma holds. The case $|X'_2| \geq (1/4 + 3\varepsilon)n$ and $X'_1 \neq \emptyset$ follows similarly.

Suppose now that $X'_2 = \emptyset$. Note that $|S| < n/2$ because the number of component in G' is at most $n - |S|$. By our assumption, $X_2 \subseteq S$, so $(1/2 - \varepsilon)n \leq |X'_1| \leq |S| \leq n/2$. Thus $G[X'_1]$ consists of at least $(1/2 - \varepsilon)n$ components, in particular it contains an independent set of size at least $(1/2 - \varepsilon)n$, and the fourth condition holds.

Finally, we assume that X'_1, X'_2 are non-empty and $|X'_1|, |X'_2| \leq (1/4 + 3\varepsilon)n$. It follows that $|S| \geq (1/2 - 6\varepsilon)n$ and G' contains at least $|S|$ odd components, in particular $X'_1 \cup X'_2$ contains an independent set A of size at least $(1/2 - 6\varepsilon)n$. Denote $A_i = X'_i \cap A$. Then $|A_i| \geq (1/4 - 9\varepsilon)n$.

□

7.3 Hamilton cycles in bipartite graphs

The following result is a stability version of a special case of Chvátal's theorem, Theorem 7.1, which we shall use in Section 15, for the proof of Lemma 7.13 below.

Lemma 7.10. *Let $\frac{1}{n} \ll \varepsilon \ll 1$ and let G be a balanced bipartite graph on n vertices with bipartition $\{X_1, X_2\}$. Suppose that $\delta(G) \geq (1/4 - \varepsilon)n$ and between every two subsets of X_1 and X_2 of size at least $(1/4 - 3\varepsilon)n$ there are at least εn^2 edges. Then G is Hamiltonian. Furthermore, there is a Hamilton path between every pair of points x_1, x_2 where $x_1 \in X_1$ and $x_2 \in X_2$.*

We prove this result by a relatively simple application of the absorbing method of Rödl, Ruciński and Szemerédi [22]. In fact, all we need for this proof is Lemma 5.5, asserting the existence of short absorbing paths in robust subgraphs, and the Regularity Lemma. A graph G as in Lemma 7.10 is $(\varepsilon, 2)$ -weakly robust, thus by Lemma 5.5, it is possible to find an absorbing path P in G . We consider the reduced graph Γ , obtained from applying the Regularity Lemma on the graph $G \setminus V(P)$. We deduce from the conditions of the lemma that Γ has an almost perfect matching, implying that G contains a cycle extending P and spanning almost all vertices. The remaining vertices may be absorbed by P .

Proof of Lemma 7.10. It is easy to check from the conditions that G is $(\varepsilon, 2)$ weakly robust with bipartition $\{X_1, X_2\}$. Indeed, let $x_1 \in X_1, x_2 \in X_2$. Recall that $|\text{con}_{G,2}(x_1, x_2)|$ is the number of paths of length three in G between x_1 and x_2 . Then $|\text{con}_{G,2}(x_1, x_2)|$ is the number of edges between $N(x_1) \setminus \{x_2\}$ and $N(x_2) \setminus \{x_1\}$, which, by the assumptions in the lemma is at least εn^2 .

It follows from Lemma 5.5, that for $\rho > 0$ is small enough, there exists a $\rho^2 n$ -absorbing path P in G of length at most ρn . Namely, if $W \subseteq V(G) \setminus V(P)$ is such that $|W \cap X_1| = |W \cap X_2| \leq \rho^2 n$, then $G[V(P) \cup W]$ contains a Hamilton path with the same ends as P . Note that we may assume for convenience that P has one end in X_1 and the other in X_2 .

Pick $\eta > 0$ suitably small. Apply the regularity lemma, Lemma 6.1, with the graph $G \setminus V(P)$, the bipartition $\{X'_1, X'_2\}$, where $X'_i = X \setminus V(P)$ and parameter η . Let G' be the subgraph of G promised by Lemma 6.1 and let $\{V_0, \dots, V_m\}$ be the given partition. Denote by m_i the number of parts V_j (where $j \geq 1$) which are contained in X_i (so $m = m_1 + m_2$). Without loss of generality, we assume that $m_1 \geq m_2$.

Claim 7.11. $m_1 - m_2 \leq 2\eta m$.

Proof. Denote by $n' = |V(G')| = n - |P|$. Recall that $|V_0| \leq \eta n'$, and $|V_1| = \dots = |V_m| \geq \frac{(1-\eta)n'}{m}$. Since $|X'_1| = |X'_2|$, we obtain $(m_1 - m_2)|V_1| \leq |V_0| \leq \eta n'$. It follows that $m_1 - m_2 \leq \frac{\eta n'}{|V_1|} \leq \frac{\eta m}{1-\eta} \leq 2\eta m$. \square

Let Γ be the $(\eta, 4\eta)$ -reduced graph defined by the m_2 clusters V_i contained in X_2 and by some m_2 clusters contained in X_1 . Denote by $\{Y_1, Y_2\}$ the bipartition of Γ .

Claim 7.12. Γ is a balanced bipartite graph on $m' = 2m_2$ vertices with $\delta(\Gamma) \geq (1/4 - 2\varepsilon)m'$. Furthermore, for every choice of subsets $A_i \subseteq Y_i$, where $|A_1|, |A_2| \geq (1/4 - 2\varepsilon)m$, we have $e(\Gamma[A_1, A_2]) \geq \frac{\varepsilon}{2}(m')^2$.

Proof. Both parts of the claim follow from Condition 4 in Lemma 6.1 and the definition of Γ as long as η and ρ are small enough. \square

It follows from Lemma 7.8, that Γ contains a perfect matching, which is a connected matching as Γ is connected. By Lemma 6.9, we obtain a cycle C in G , containing the path P and spanning all but at most $6\eta n \leq \rho^2 n$ vertices. Denote $W = V(G) \setminus V(C)$. Since G is a balanced bipartite graph, we have $|W \cap X_1| = |W \cap X_2| \leq \rho^2 n$. It follows from the absorbing property of P that the vertices of W may be absorbed into P and thus into C to obtain a Hamilton cycle.

It is easy to modify the proof to obtain a Hamilton path between any given vertices $x_i \in X_i$. \square

7.4 Monochromatic cycle partitions in 2-coloured graphs with the red graph almost disconnected

The following two lemmas, Lemmas 7.13 and 7.14, state that a 2-coloured graph G has the desired partition into a red cycle and a blue one if G admits some restrictive structural property. We shall use these results several times in the following sections and delay their proofs to the ends of the paper. We prove Lemma 7.13 in Section 15 and Lemma 7.14 in Section 16.

Lemma 7.13. *Let $\frac{1}{n} \ll \varepsilon \ll 1$ and let G be a graph on n vertices with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that $S, T \subseteq V(G)$ satisfy the following conditions.*

- *S, T are disjoint and $|S|, |T| \geq (1/2 - \varepsilon)n$.*
- *$\delta(G_B[S, T]) \geq (1/4 - \varepsilon)n$.*
- *For every $S' \subseteq S, T' \subseteq T$ with $|S'|, |T'| \geq (1/4 - 100\varepsilon)n$, we have $e(G_B[S', T']) \geq 25\varepsilon n^2$.*

Then $V(G)$ may be partitioned into a red cycle and a blue one.

Lemma 7.14. *Let $\frac{1}{n} \ll \varepsilon \ll 1$ and let G be a graph on n vertices with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that there exists a partition $\{S, T, X\}$ of $V(G)$ with the following properties.*

- *$|S|, |T| \geq (1/2 - \varepsilon)n$.*
- *$|X| \leq 2$ and if $|X| = 2$, there exists $u \in X$ such that $\deg_R(x, S) \leq \varepsilon n$ or $\deg_R(x, T) \leq \varepsilon n$.*
- *The sets S and T belong to different components of $G_R \setminus X$.*

Then $V(G)$ may be partitioned into a red cycle and a blue one.

This concludes the preliminary material needed for the proof of Theorem 1.1. We are now finally ready to turn to the heart of the proof.

8 Rough structure

In this section we make the first step towards our proof of Theorem 1.1. We use the Regularity Lemma, Lemma 6.1, to obtain information about the rough structure.

Lemma 8.1. *Let $\frac{1}{n} \ll \alpha, \frac{1}{k} \ll \varepsilon \ll 1$ and let G be a graph with $\delta(G) \geq 3n/4$. Let $E(G) = E(G_B) \cup E(G_R)$ be a 2-colouring of G . Then one of the following assertions holds, where a robust component refers to an (α, k) -robust component, possibly with the roles of red and blue reversed.*

1. *There exists a monochromatic strongly robust blue component on at least $(1 - \varepsilon)n$ vertices.*
2. *There exists a weakly robust blue component of order at least $(1 - \varepsilon/4)n$ and a red strongly robust component of order at least $(1/2 + \varepsilon/2)n$.*
3. *There exists a weakly robust blue component with bipartition $\{X_1, X_2\}$ where $|X_1|, |X_2| \geq (1/2 - \varepsilon)n$ and for each $i \in [2]$, $e(G_B[X_i]) \leq \varepsilon n^2$ and one of the following holds.*
 - (a) *$G_R[X_i]$ is strongly robust.*
 - (b) *$G_R[X_i]$ is weakly robust with bipartition $\{Y_{i,1}, Y_{i,2}\}$ satisfying $|Y_{i,j}| \geq (1/4 - \varepsilon)n$ and $e(G_R[Y_{i,j}]) \leq \varepsilon n^2$.*

- (c) *There exists a partition $\{Y_{i,1}, Y_{i,2}\}$ of X_i such that $|Y_{i,j}| \geq (1/4 - \varepsilon)n$, $G_R[Y_{i,j}]$ is strongly robust and $e(G_R[Y_{i,1}, Y_{i,2}]) \leq \varepsilon n^2$.*

Furthermore, if Condition (3b) holds for $i \in [2]$, then in addition $e(G_R[Y_{1,j}, Y_{2,j}]) \leq \varepsilon n^2$ for $j \in [2]$.

4. *There exist a blue strongly robust component and a red robust component, each has order at least $(3/4 - \varepsilon)n$ and together the span all but at most εn of the vertices.*
5. *There exist sets X_1, X_2, Y_1, Y_2 of order at least $(1/2 - \varepsilon)n$ such that*
 - X_1, X_2 are disjoint, Y_1, Y_2 are disjoint and $X_1 \cup X_2 = Y_1 \cup Y_2$.
 - $|X_i \cap Y_j| \geq (1/4 - \varepsilon)n$ for $i, j \in [2]$.
 - $G_B[X_i]$ is robust and $G_R[Y_i]$ is strongly robust.

We remark that in light of the variety of extremal examples for Theorem 1.1 (see Section 2), it should not be surprising that there is a large number of cases to consider for the rough structure. Furthermore, it is perhaps useful to note that many of the above cases describe the structure of the extremal examples we gave in Section 2. For example, the left-hand graph in Figure 1 corresponds to Case 5, Figures 2 and 3 correspond to Case 3 and Figure 4 corresponds to Case 4.

This lemma, technical as it seems, is a simple application of Lemmas 6.3 and 6.7, which imply that monochromatic components in the reduced graph correspond to robust components in the original graph. Before proving Lemma 8.1, we give a brief overview of the proof. After applying the Regularity Lemma with suitable parameters, we obtain a reduced graph Γ , which has minimum degree close to $3m/4$ where $m = |\Gamma|$. It is a routine check to verify that either there is a spanning monochromatic component, or there are two monochromatic components of size almost $3m/4$ spanning the whole vertex set, or for each colour there are two almost half-sized components spanning the whole vertex set. In the case where there is a spanning monochromatic component, further analysis is needed to show that one of the cases (1, 2, 3) holds.

Proof. Set $\eta = 48\varepsilon$. Let Γ be a $(\eta, 6\eta)$ -reduced graph obtained by applying Lemma 6.1 to the graph G . Note that $\delta(\Gamma) \geq (3/4 - 13\eta)m$, where $m = |\Gamma|$. Without loss of generality, we assume that the largest monochromatic component is blue and denote it by Φ_1 .

Suppose first that Φ_1 is a spanning subgraph of Γ . If it is in addition non-bipartite, by Lemma 6.3, there is a strongly robust blue component F_1 of order at least $(1 - 2\eta)n$, as in (1).

Thus we assume that Φ_1 is bipartite, with bipartition $\{X_1, X_2\}$ where $|X_1| \geq |X_2|$. If $|X_1| > (1/2 + 26\eta)m$, then $\delta(\Gamma[X_1]) \geq (3/4 - 13\eta)m - |X_2| > |X_1|/2$. It follows that $\Gamma[X_1]$ is a red non bipartite component. By Lemma 6.7, we obtain a weakly robust blue component F_1 on at least $(1 - 3\eta)n$ vertices and a red strongly robust component F_2 on at least $(1/2 + 23\eta)n$ vertices, as in (2).

We assume now that $|X_1| \leq (1/2 + 26\eta)m$. Then $(1/2 - 26\eta)m \leq |X_1|, |X_2| \leq (1/2 + 26\eta)m$. Denote $\Gamma_i = \Gamma[X_i]$ for $i \in [2]$. Γ_i contains only red edges and $\delta(\Gamma_i) \geq (1/4 - 39\eta)m$. Then one of the following holds for $i \in [2]$.

1. Γ_i is connected in red and non-bipartite.

2. Γ_i is connected and bipartite. Furthermore, it has minimum degree at least $(1/4 - 39\eta)m$.
3. Γ_i consists of two red components, each of order at least $(1/4 - 39\eta)m$.

By the definition of a reduced graph and our choice of parameters, the number of blue edges of G which are not present in G' is at most $14\eta n^2$ and similarly for the red edges. It follows from Lemma 6.7 that one of the conditions in (3) holds. Suppose that Condition (2) holds for both Γ_1 and Γ_2 . Denote the bipartition of Γ_i by $\{Y_{i,1}, Y_{i,2}\}$. If there are no edges between $Y_{1,j}$ and $Y_{2,j}$ for $j = 1, 2$, Condition (3) is satisfied. Otherwise, without loss of generality, there is an edge between $Y_{1,1}$ and $Y_{2,1}$. Then the red graph Γ_B is connected. But we assumed that there Γ has no spanning non-bipartite monochromatic component, so there are no edges between $Y_{i,1}$ and $Y_{3-i,2}$ for $i \in [2]$. Thus, up to relabelling of $Y_{i,j}$, Condition (3) holds.

We assume now that Φ_1 does not span Γ . Denote $s = |\Phi_1|$. Suppose first that $s > (1/2 + 26\eta)m$. Denote $U = V(\Gamma) \setminus V(\Phi_1)$. Then every vertex $u \in U$, is incident to at least $(3/4 - 13\eta)m - (m - s) > s/2$ neighbours. It follows that every two vertices outside of $V(\Phi_1)$ have a common red neighbour, implying that U is contained in a red component Φ_2 of order at least $(3/4 - 13\eta)m$. Indeed, pick $u \in U$. Then the red neighbourhood of u in Γ is contained in Φ_2 as well as U .

By the choice of Φ_1 as the largest connected monochromatic subgraph, we have that $|\Phi_1| \geq (3/4 - 13\eta)m$. The components Φ_1, Φ_2 cover Γ and intersect in at least $(1/2 - 26\eta)n$ vertices. By the following claim, at least one of Φ_1, Φ_2 is non-bipartite, implying that condition (4) holds, by Lemma 6.7.

Claim 8.2. *At least one of the graphs Φ_1, Φ_2 is not bipartite.*

Proof. Suppose otherwise. Denote $X_i = V(\Phi_i) \setminus V(\Phi_{3-i})$ for $i \in [2]$ and $Y = V(\Phi_1) \cap V(\Phi_2)$. Then for every $x_1 \in X_1, x_2 \in X_2$, the vertices x_1, x_2 are non-adjacent in Γ . Thus, x_1 sends at least $(3/4 - 13\eta)m - |X_1|$ (blue) edges to Y and similarly x_2 sends at least $(3/4 - 13\eta)m - |X_2|$ red edges into Y . If both Φ_1 and Φ_2 are bipartite, it follows that Y contains a set A_1 of $(3/4 - 13\eta)m - |X_1|$ vertices spanning no blue edges and a set A_2 of $(3/4 - 13\eta)m - |X_2|$ spanning no red edges. It follows that Y contains an independent set of size at least $|A_1| + |A_2| - |Y| \geq (3/2 - 26\eta)m - (|X_1| + |X_2| + |Y|) \geq (1/2 - 26\eta)m$. This is a contradiction to the minimum degree condition on Γ . \square

It remains to consider the case where $s = |\Phi_1| \leq (1/2 + 26\eta)m$. An argument similar to a previous one shows that if $s < (1/2 - 26\eta)m$, every two vertices of Φ_1 have a common red neighbour outside of Φ_1 , contradicting the choice of Φ_1 as the largest monochromatic component. Thus we have that $(1/2 - 26\eta)m \leq s \leq (1/2 + 26\eta)m$. Note that we may find $u_1, u_2 \in V(\Phi_1)$ which have no common red neighbour outside of Φ_1 (otherwise there is a red component of order larger than $|\Phi_1|$ contradicting our choice of Φ_1). Denote by X_i the set of red neighbours of u_i outside of $V(\Phi_1)$. Let Y_i be the red neighbourhood of X_i in Φ_1 . It follows from the minimum degree condition and the order of Φ_1 that $|X_i|, |Y_i| \geq (1/4 - 39\eta)m$. Furthermore, the sets X_1, X_2, Y_1, Y_2 are disjoint, there are no red edges in between $X_1 \cup Y_1$ and $X_2 \cup Y_2$ and no blue edges between $X_1 \cup X_2$ and $Y_1 \cup Y_2$. In particular, there are no edges between X_i and Y_{3-i} . Considering the minimum degree conditions and the size of the various sets, it follows that the blue subgraph $\Phi_2 = \Gamma_B \setminus V(\Phi_1)$ is connected. Similarly, $\Gamma_R[X_i \cup Y_i]$ is connected. Moreover, of the four components, there cannot be both a red and a blue bipartite component. Condition (5) follows. \square

Proof of the main theorem

We now prove Theorem 1.1, using Lemma 8.1 and other results which we shall state and prove in subsequent sections.

Theorem (1.1). *There exists n_0 such that if a graph on $n \geq n_0$ vertices and minimum degree at least $3n/4$ is 2-coloured then its vertex set may be partitioned into two monochromatic cycles of different colours.*

Proof of Theorem 1.1. Let $\frac{1}{n} \ll \varepsilon \ll 1$ and let G be a graph on $n \geq n_0$ vertices with minimum degree at least $3n/4$ and a red and blue colouring of the edges. By Lemma 8.1, we may assume that one of the Cases (1 - 5) from the statement of the lemma hold. It remains to conclude that in each of these cases, we may find a partition of $V(G)$ into a red cycle and a blue one. We prove this for each of the above cases using lemmas appearing in Sections (9 - 14).

We start by resolving Case (2), which is perhaps the easiest to deal with, in Lemma 9.1 in Section 9.

Case (3) is dealt with in Sections 10, 11. Lemmas 10.1, 10.2 and 10.3 share some similarities and are used to prove all possible combinations of Cases (3a, 3b, 3c) except for the case where Condition (3b) holds for both graphs in question. The proof of Theorem 1.1 in the latter case can be deduced from Lemma 11.1 and is of different nature.

Case (4) can be resolved by Lemma 12.1 in Section 12. It shares some ideas with the proofs in Section 10, but requires further analysis. Case (5) is dealt with by Lemma 13.1 in Section 13. Case (1) turns out to be hardest to deal with, thus we prove it last in Lemma 14.1 in Section 14. \square

9 Large weakly robust blue component

In this section we deal with Case (2) of Lemma 8.1. This is perhaps the easiest case to deal with. We recommend the reader to follow the proof here carefully, since the methods appearing here will be used in later sections, often in less detail.

In order prove Theorem 1.1 in Case (2), we prove the following lemma. Note that we make the further assumption that the given robust components cover all the vertices of G (rather than almost all of them). This can be easily be justified by Lemma 5.4, which states that given a robust subgraph F , the graph obtained by adding vertices of large degree into F remains robust.

Lemma 9.1. *Let $\frac{1}{n} \ll \varepsilon, \alpha, \frac{1}{k} \ll 1$ and let G be a graph of order n with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that F_1, F_2 satisfy the following conditions.*

- $F_1 \subseteq G_B$ is (α, k) weakly robust with bipartition $\{X, Y\}$ and $|F_1| \geq (1 - \varepsilon)n$.
- $F_2 \subseteq G_R$ is (α, k) -strongly robust and $|F_2| \geq (1/2 + 2\varepsilon)n$.
- $V(G) = V(F_1) \cup V(F_2)$.

Then $V(G)$ may be partitioned into a blue cycle and a red cycle.

This case is the most straightforward of the various cases arising from Lemma 8.1. We apply Lemma 5.5 to find vertex disjoint absorbing paths P_i in F_i for $i \in [2]$. Then, using the regularity lemma and the connected matching method, we find two vertex-disjoint monochromatic cycles containing the paths P_1, P_2 and covering almost all of the vertices. Finally, we use the absorption property to insert the remaining vertices into the paths P_1, P_2 so as to obtain the desired monochromatic cycle partition.

Proof of Lemma 9.1. We use Lemma 5.5 to build absorbing paths for F_1 and F_2 . More precisely, given a suitably small $\rho > 0$, there exists vertex disjoint paths $P_i \subseteq F_i$ for $i \in [2]$ satisfying the following conditions, where $U = V(P_1) \cup V(P_2)$.

- $|P_i| \leq \rho n$.
- For every set $W \subseteq V(F_1) \setminus U$, with $|W \cap X| = |W \cap Y| \leq \rho^2 n$, the graph $F_1[V(P_1) \cup W]$ has a Hamilton path with the same ends as P_1 .
- For every set $W \subseteq V(F_2) \setminus U$ of size at most $\rho^2 n$, the graph $F_2[V(P_2) \cup W]$ has a Hamilton path with the same ends as P_2 .

Indeed, by Lemma 5.5, we may find a $\rho^2 n$ -weakly-absorbing path P_1 in F_1 of length at most ρn . By Lemma 5.2, the subgraph $F'_2 = F_2 \setminus V(P_1)$ is robust (with suitable parameters), thus we may find a $\rho^2 n$ absorbing path P_2 in F'_2 of length at most ρn .

Using the regularity lemma, we shall find vertex-disjoint monochromatic cycles, containing the paths P_1 and P_2 and covering most vertices of G . We then cover the remaining vertices using the absorption properties of the paths P_1 and P_2 . However, the fact that P_1 is weakly-absorbing presents a technical difficulty which we overcome as follows.

Recall that $\{X, Y\}$ is the bipartition of F_1 . Without loss of generality, suppose that $|X \cap V(F_2)| \geq n/8$. Note that $|Y| \geq \alpha n$ by the minimum degree condition on $F_1[X, Y]$. Pick subsets $A_1 \subseteq X \cap V(F_2)$ and $A_2 \subseteq Y$, disjoint of U , such that $|A_1| = \rho^2 n/2$ and $|A_2| = \rho^2 n/4$, and denote $A = A_1 \cup A_2$.

Apply the Regularity Lemma, Lemma 6.1, to the graph $G \setminus (U \cup A)$ with parameter $\eta = \rho^2/16$ and the cover $\{V(F_1), V(F_2)\}$. Let Γ be the corresponding $(\eta, 6\eta)$ -reduced graph. Note that $\delta(G \setminus (U \cup A)) \geq (3/4 - 3\rho)n$. It follows from observation 6.2 that for a suitable choice of ρ we have $\delta(\Gamma) \geq (3/4 - \varepsilon/2)m$, where $m = |\Gamma|$.

The blue subgraph of Γ determined by the clusters contained in $V(F_1)$ is connected by Lemma 6.8. Let Φ_1 be the connected component of Γ_B containing that subgraph. Similarly, let Φ_2 be the connected component of Γ_R containing the clusters which are contained in $V(F_2)$. Note that Φ_1 and Φ_2 cover $V(\Gamma)$. Furthermore, $|\Phi_1| \geq (1 - 3\varepsilon/2)m$ and $|\Phi_2| \geq (1/2 + 3\varepsilon/2)m$. Let Γ' be the union of these graphs.

Claim 9.2. Γ' has a perfect matching.

Proof. Denote $V_1 = V(\Phi_1) \cap V(\Phi_2)$ and $V_2 = V(\Phi_1) \Delta V(\Phi_2)$. Note that V_1, V_2 partition $V(\Gamma)$ and $|V_1| \geq m/2$.

Recall that $\delta(\Gamma) \geq (3/4 - \varepsilon/2)m$. It follows that for every $v \in V_1$, we have $\deg_{\Gamma'}(v) \geq (3/4 - \varepsilon/2)m$ because all edges of Γ incident to v are in Γ' . Vertices not in Φ_1 have blue degree at most $\frac{3}{2}\varepsilon m$. Similarly, vertices not in Φ_2 have red degree at most $(1/2 - 3\varepsilon/2)m$. It follows that for every $v \in V_2$, we have $\deg_{\Gamma'}(v) \geq (1/4 + \varepsilon)m$.

We conclude from Theorem 7.1 that Γ has a Hamilton cycle and in particular a perfect matching. \square

Remark. In the last claim, we implicitly assumed that m is even. It is indeed possible to make this further assumption in the Regularity Lemma. We shall make this assumption whenever convenient without stating so explicitly.

By Claim 9.2, Γ has a perfect matching consisting of a blue connected matching in Φ_1 and a red connecting matching in Φ_2 . Thus, we may use Lemma 6.9 to obtain a blue cycle C_1 and a red cycle C_2 which are disjoint, each C_i contains the respective absorbing path P_i and together they cover all but at most $7\eta n \leq \rho^2 n/4$ vertices of $V(G) \setminus A$.

We now show how to absorb the leftover vertices into the cycles C_1, C_2 . Let B be the set of vertices which are not contained in the cycles C_1, C_2 or in the set A (so $|B| \leq \rho^2 n/4$). Denote $B_1 = X \cap B$, $B_2 = Y \cap B$, and $B_3 = B \setminus V(F_1)$. Recall that $A_1 \subseteq X \cap V(F_2)$ and $A_2 \subseteq Y$ are disjoint of the cycles C_1, C_2 and have sizes $\rho^2 n/2$ and $\rho^2 n/4$ respectively. It follows that $\rho^2 n/4 \leq |A_2 \cup B_2| \leq \rho^2 n/2$. Thus we may choose $A'_1 \subseteq A_1$ such that $A'_1 \cup B_1$ and $A_2 \cup B_2$ are of equal size which is most $\rho^2 n/2$. Recall that the path P_1 , which is contained in C_1 is $\rho^2 n$ -absorbing in F_1 , so these sets can be absorbed by P_1 and thus by C_1 . We remain with the vertices $(A_1 \setminus A'_1) \cup B_3$. There are at most $\rho^2 n$ of them and they belong to F_2 , so we may absorb them into P_2 and thus into C_2 . This completes a partition of $V(G)$ into a red and a blue cycle. \square

Now that we have proved Theorem 1.1 in Case (2) of Lemma 8.1, we are ready to consider harder cases.

10 Large weakly robust blue graph with almost balanced bipartition

In this section we consider Condition (3) of Lemma 8.1, where we have a large weakly robust blue component with an almost balanced bipartition $\{X_1, X_2\}$. There are three possibilities for each of the red graphs $G_R[X_i]$. In this section we focus on the case where at least one of these red subgraphs satisfies conditions (3a) or (3c). The remaining case, with both red graphs $G_R[X_1], G_R[X_2]$ satisfying condition (3b) requires a proof of different nature and is thus dealt with in the following section.

The main idea in the various cases arising here is that if two red robust components may be joined by two vertex-disjoint red paths, then they can essentially be treated as one bigger component, at which case we may use the argument of the previous section to finish the proof. If that is not possible, we deduce that the red graph G_R may be disconnected by removing a small number of vertices into two almost half-sized subgraphs. This case may be resolved by Lemmas 7.13 and 7.14.

The following lemma deals with the case where one of the red graphs in question satisfies condition (3a) and the other satisfies (3a) or (3b).

Lemma 10.1. *Let $\frac{1}{n} \ll \varepsilon, \alpha, \frac{1}{k} \ll 1$ and let G be a graph of order n with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that F_1, F_2, F_3 satisfy the following assertions.*

- $F_1 \subseteq G_B$ is (α, k) weakly robust and with bipartition $\{X, Y\}$, where $|X|, |Y| \geq (1/2 - \varepsilon)n$ and $e(G_B[X]), e(G_B[Y]) \leq \varepsilon n^2$.
- $F_2 \subseteq G_R$ is (α, k) -strongly robust and $V(F_2) = X$.
- $F_3 \subseteq G_R$ is (α, k) -robust with $V(F_3) = Y$.

Then $V(G)$ may be partitioned into a blue cycle and a red cycle.

To prove this lemma, we show that we may either join the two robust components F_2, F_3 or we can finish using Lemma 7.13.

Proof of Lemma 10.1. By Menger's theorem, one of the following holds.

1. There exist two vertex-disjoint red paths P_2, P_3 , each having one end in F_2 and the other in F_3 .
2. There exists $u \in V(G)$ such that $F_2 \setminus \{u\}$ and $F_3 \setminus \{u\}$ are disconnected in $G_R \setminus \{u\}$.

In Case (2), it is easy to deduce that the required monochromatic cycle partition exists from Lemma 7.14.

It remains to consider Case (1). The idea is to use the paths P_2 and P_3 so as to essentially connect the two components F_2, F_3 into one large component. We achieve this as follows.

Note that we may assume that the internal vertices of P_2, P_3 belong to $V(G) \setminus (V(F_2) \cup V(F_3))$. It follows that $|P_2|, |P_3| \leq 2\varepsilon n$ and the components F_1, F_2, F_3 remain robust after removing the vertices of the paths P_2, P_3 . Note that every vertex has either large (say, at least αn) blue degree into F_1 , or large red degree into either F_2 or F_3 . It follows that we may extend these components to cover the remaining vertices. Namely, using Lemma 5.4, we obtain $(\alpha/2, k+2)$ -robust components F'_1, F'_2, F'_3 which extend the given components and cover $V(G) \setminus (V(P_2) \cup V(P_3))$. Denote by $\{X', Y'\}$ the bipartition of F'_1 .

By Lemma 5.5, F'_i contains a $\rho^2 n$ -absorbing path P'_i of length at most ρn for each $i \in [3]$. We may assume that the paths P'_1, P'_2, P'_3 are vertex disjoint. For $i \in \{2, 3\}$, we may connect P_i with P'_i in F_i to a path Q_i using at most $k+2$ additional vertices. For convenience, we denote $Q_1 = P'_1$. To conclude, the paths $Q_1 \subseteq G_B$ and $Q_2, Q_3 \subseteq G_R$ are vertex-disjoint paths, such that Q_i is $\rho^2 n$ -absorbing in F'_i . Furthermore, each of Q_2, Q_3 has one end in F'_2 and one in F'_3 .

Recall that F_1 is weakly robust and that F_2 is strongly robust. It is perhaps not evident from the definitions that a strongly robust graph is weakly robust, but for our purpose, the place where the difference is important is in the definition of an absorbing path. But clearly, a weakly absorbing path is strongly robust. Thus, it suffices to consider the case where F'_3 is weakly robust with bipartition $\{Z_1, Z_2\}$.

Similarly to the proof in Section 9, in order to overcome the technical issues arising when dealing with weakly robust components, we pick sets A_1, A_2, A_3 such that

- A_1, A_2, A_3 are pairwise vertex disjoint and do not intersect $V(Q_1) \cup V(Q_2) \cup V(Q_3)$.
- $A_1 \subseteq Z_1 \cap Y$, $A_2 \subseteq Z_2 \cap Y$ and $A_3 \subseteq X \cap V(F_2)$.
- $|A_1| = \rho^2 n/16$, $|A_2| = \rho^2 n/4$ and $|A_3| = \rho^2 n/2$.

We now consider the $(\eta, 6\eta)$ -reduced graph obtaining by applying the Regularity Lemma, Lemma 6.1, to the graph obtained from G by removing the vertices in the Q_i 's and A_i 's (with the cover $\{V(F'_1), V(F'_2), V(F'_3)\}$). The reduced graph Γ consists of a large blue component Φ_1 (containing almost all vertices) and two disjoint almost half-sized connected red subgraphs Φ_2, Φ_3 . It is easy to verify, similarly to the proof of Claim 9.2, using Theorem 7.1, that Γ has a perfect matching consisting of edges in Φ_1, Φ_2 and Φ_3 .

By Lemma 6.9, there exist a blue cycle C_1 and a red cycle C_2 such that

- C_1 and C_2 are vertex-disjoint and do not intersect $A = A_1 \cup A_2 \cup A_3$.
- They cover all but at most $4\eta n$ vertices of $V(G) \setminus A$.
- C_1 contains the path Q_1 and C_2 contains the paths Q_2, Q_3 .

Let us elaborate slightly more on how to obtain the required cycles by pointing out that C_2 may be obtained by connecting the ends of Q_2, Q_3 by two paths, one in F'_2 and the other in F'_3 .

Let B be the set of vertices which do not belong to the cycles C_1, C_2 or to A (so $|B| \leq 7\eta n \leq \rho^2 n/16$) and denote

$$\begin{aligned} B_1 &= B \cap Z_1, & B_2 &= B \cap Z_2 \\ B_3 &= (B \cap X') \setminus (B_1 \cup B_2) \\ B_4 &= (B \cap Y') \setminus (B_1 \cup B_2) \\ B_5 &= B \setminus (B_1 \cup \dots \cup B_4). \end{aligned}$$

We perform the following steps in order to absorb B .

- We have $|B_2| \leq \rho^2 n/16 \leq |A_1 \cup B_1| \leq \rho^2 n/8$ and $|A_2| = \rho^2 n/4$. Thus we may choose $A'_2 \subseteq A_2$ such that $|A_1 \cup B_1| = |A'_2 \cup B_2|$. The vertices $A_1 \cup B_1 \cup A'_2 \cup B_2$ can be absorbed into Q_3 .
- Similarly, we may choose $A'_3 \subseteq A_3$ such that $|(A_2 \setminus A'_2) \cup B_3| = |A'_3 \cup B_4|$. The vertices $(A_2 \setminus A'_2) \cup B_3 \cup A'_3 \cup B_4$ can be absorbed into Q_1 .
- The remaining vertices $(A_3 \setminus A'_3) \cup B_5$ can be absorbed into Q_2 .

This completes the partition of $V(G)$ into a blue cycle and a red one. □

The following lemma deals with the case where one of the red graphs in question satisfies Condition (3c) in Lemma 8.1 and the other satisfies one of the other two conditions.

Lemma 10.2. *Let $\frac{1}{n} \ll \varepsilon, \alpha, \frac{1}{k} \ll 1$ and let G be a graph of order n with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that F_1, F_2, F_3, F_4 satisfy the following assertions.*

- $F_1 \subseteq G_B$ is (α, k) weakly robust and with bipartition $\{X, Y\}$, where $|X|, |Y| \geq (1/2 - \varepsilon)n$.
- $F_2 \subseteq G_R$ is (α, k) -robust and $V(F_2) = X$.
- $F_3, F_4 \subseteq G_R$ are (α, k) strongly robust components of order at least $(1/4 - \varepsilon)n$ whose vertex sets partition Y . Furthermore, $e(G_R[V(F_3), V(F_4)]) \leq \varepsilon n^2$.

Then $V(G)$ may be partitioned into a blue cycle and a red cycle.

The proof of this lemma is similar to the previous one. Either F_2 may be joined to one of F_3, F_4 or we may finish using Lemma 7.13.

Proof of Lemma 10.2. Similarly to the previous case, two possibilities arise.

1. There exist vertex-disjoint red paths P_1, P_2 with one end in F_2 and either both have the other end in F_3 or both have the other end in F_4 .
2. There exists a set U of size at most 2 such that $F_1 \setminus U$ and $(F_2 \cup F_3) \setminus U$ are disconnected in $G_R \setminus U$.

In Case (2), the required monochromatic cycle partition exists by Lemma 7.13. Indeed, we may find $S \subseteq X \setminus U$ and $T \subseteq Y \setminus U$ such that the three conditions in the lemma hold, for some parameter $\eta = \eta(\varepsilon)$. In particular, the third condition holds because most vertices in Y have degree at most $(1/4 + \sqrt{\varepsilon})n$ in $G[Y]$, and thus have degree at least $(1/2 - \sqrt{\varepsilon})n$ into X .

Case (1) may be dealt with similarly to the previous proof of Lemma 10.1, we omit further details. \square

The following lemma deals with the remaining case, where both red graphs satisfy condition (3c) in Lemma 8.1.

Lemma 10.3. *Let $\frac{1}{n} \ll \varepsilon, \alpha, \frac{1}{k} \ll 1$ and let G be a graph of order n with $\delta(G) \geq 3n/4$ and with a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that F_1, F_2, F_3, F_4, F_5 satisfy the following conditions.*

- $F_1 \subseteq G_B$ is (α, k) weakly robust and with bipartition $\{X, Y\}$, where $|X|, |Y| \geq (1/2 - \varepsilon)n$.
- $F_2, F_3 \subseteq G_B$ are (α, k) strongly robust components of order at least $(1/4 - \varepsilon)n$ whose vertex sets partition X .
- $F_4, F_5 \subseteq G_B$ are (α, k) strongly robust components of order at least $(1/4 - \varepsilon)n$ whose vertex sets partition Y .
- $e(G_R[V(F_2), V(F_3)]), e(G_R[V(F_4), V(F_5)]) \leq \varepsilon n^2$.

Then $V(G)$ may be partitioned into a blue cycle and a red cycle.

To prove this lemma we follow similar ideas to the previous results in this section. We show that either at least three of the four components F_2, F_3, F_4, F_5 may be joined or we may finish using Lemma 7.13 or Lemma 7.14.

Proof of Lemma 10.3. We consider four cases. In order to be able to distinguish between them, we define a graph H on vertex set $\{2, 3, 4, 5\}$ with an edge (i, j) , where $i \in \{2, 3\}, j \in \{4, 5\}$ if $e(G_R[F_i, F_j]) \geq \varepsilon n^2$ (so H is a bipartite graph on four vertices with bipartition $\{[2, 3], [4, 5]\}$). Clearly, one of the following conditions.

1. H contains a path of length 2.
2. H consist of two vertex-disjoint edges.
3. H has exactly one edge.
4. H is the empty graph.

In Case (1), without loss of generality suppose that $(2, 4), (2, 5) \in E(H)$. This means that $e(G_R[F_2, F_4]), e(G_R[F_2, F_5]) \geq \varepsilon n^2$. In particular, there exist four vertex-disjoint edges $e_1, e_2 \in G_R[F_2, F_4], e_3, e_4 \in G_R[F_2, F_5]$. We deduce that the components F_2, F_4, F_5 may be joined to form a red component of size at least $(3/4 - 3\varepsilon)n$. We may now finish the proof of Lemma 10.3 similarly to the previous lemmas in this section.

We now suppose that Case (2) holds. Without loss of generality, $E(H) = \{(2, 4), (3, 5)\}$. If there are two vertex-disjoint red paths P_1, P_2 between $V(F_2) \cup V(F_4)$ and $V(F_3) \cup V(F_5)$, we conclude that the four component F_2, F_3, F_4, F_5 may be joined (note that to join F_2 to F_4 and F_3 to F_5 we use edges between them which can be chose so as to not intersect the given paths P_1, P_2). Otherwise, there is a vertex $u \in V(G)$ such that $G_R \setminus \{u\}$ is disconnected, with F_2, F_4 in one component and F_3, F_5 in another. The proof of Lemma 10.3 can be completed by Lemma 7.14.

We now assume that Case (3) holds. Without loss of generality, $e(G_R[V(F_2), V(F_4)]) \geq \varepsilon n^2$. Consider the graph $G' = G_R[V(F_2) \cup V(F_4), V(F_3) \cup V(F_5)]$. If it contains a matching of size at least 5, then the one of F_2, F_4 may be joined to one of F_3, F_5 . Since there are many edges between F_2 and F_4 , it follows that we may form a red component using F_2, F_4 and one of F_3, F_5 . The proof may be completed as before. Thus we assume that the above graph has no matching of size 5. It follows that we may remove four vertices from G' so as to disconnect F_2 and F_4 from F_3 and F_5 . Thus there exists a set $U \subseteq V(G)$ of size at most $3\varepsilon n$ such that F_2 and F_4 are disconnected from F_3 and F_5 . Lemma 10.3 now follows from Lemma 7.13 (note that most vertices in $V(F_3) \cup V(F_5)$ have red degree at most about $n/4$).

Finally, we consider Case (4). Denote $U = V(F_2) \cup V(F_3) \cup V(F_4) \cup V(F_5)$. Let W be the union of $V(G) \setminus U$ with the set of vertices in $V(G)$ which have degree at least $2\sqrt{\varepsilon}n$ in $G_R[V(F_2) \cup V(F_3), V(F_4) \cup V(F_5)]$. Since H is the empty graph, we have $|U| \leq 5\sqrt{\varepsilon}n$, thus we may complete the proof by Lemma 7.13. \square

In order to finish the proof of Theorem 1.1 under the assumption that Condition (3) from Lemma 8.1, we need to consider the case where both graphs in question satisfy Condition (3b). This is done in the next section, Section 11.

11 Almost balanced large weakly robust blue component and two half-sized weakly robust red components

In this section we consider Case (3) from Lemma 8.1, where both graphs $G_R[X_1]$ and $G_R[X_2]$ satisfy condition (3b). Namely, we have a large weakly robust blue component with an almost balanced bipartition $\{X_1, X_2\}$. Furthermore, $G_R[X_i]$ is weakly robust for $i \in [2]$ with an almost balanced bipartition $\{Y_{i,1}, Y_{i,2}\}$ such that $e(G_R[Y_{i,1}, Y_{i,2}]) \leq \varepsilon n^2$.

Lemma 11.1. *Let $\frac{1}{n} \ll \varepsilon \ll 1$ and let G be a graph of order n with $\delta(G) \geq 3n/4$ and with a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that there exists four disjoint sets $Y_{i,j}$, $i, j \in [2]$ with the following properties.*

- $|Y_{i,j}| \geq (1/4 - \varepsilon)n$ for $i, j \in [2]$.
- $e(G[Y_{i,j}]) \leq \varepsilon n^2$.
- $e(G_B[Y_{i,1}, Y_{i,2}]) \leq \varepsilon n^2$ for $i \in [2]$.
- $e(G_R[Y_{1,j}, Y_{2,j}]) \leq \varepsilon n^2$ for $j \in [2]$.

We notice that a graph with the given conditions has a rather specific structure. Namely, the sets $Y_{i,j}$ span few edges, whereas the graphs $G_R[Y_{i,1}, Y_{i,2}]$ and $G_B[Y_{1,j}, Y_{2,j}]$ are almost complete. By Lemma 7.13, we conclude that we may finish the proof unless say $G_R[Y_{1,1}, Y_{2,2}]$ and $G_B[Y_{1,2}, Y_{2,1}]$ are almost complete. In the latter case we construct the required partition into a red cycle and a blue one “by hand”.

Proof of Lemma 11.1. The conditions imply that for some $\eta = \eta(\varepsilon)$, we can find disjoint sets S_1, S_2, T_1, T_2 such that the following holds.

- $|S_i|, |T_i| \geq (1/4 - \eta)n$ for $i \in [2]$.
- $\delta(G_R[S_1, S_2]) \geq (1/4 - \eta)n$ and $\delta(G_R[T_1, T_2]) \geq (1/4 - \eta)n$.
- $\delta(G_B[S_i, T_i]) \geq (1/4 - \eta)n$ for $i \in [2]$.
- $e(G[S_i]), e(G[T_i]) \leq \eta n^2$.

Denote $S = S_1 \cup S_2$ and $T = T_1 \cup T_2$. Consider the graph $G_B[S, T]$. If for every $S' \subseteq S$ and $T' \subseteq T$ with $|S'|, |T'| \geq (1/4 - 200\eta)n$ we have $e(G_B[S', T']) \geq 50\eta n^2$, Lemma 7.13 implies that G may be partitioned into a blue cycle and a red one. Thus we may assume that there exist subsets $S' \subseteq S$ and $T' \subseteq T$ of size $(1/4 - 200\eta)n$ such that $e(G_B[S', T']) \leq 50\eta n^2$.

Claim 11.2. *Either $|S_1 \cap S'|, |T_2 \cap T'| \leq 10\sqrt{\eta}n$ or $|S_2 \cap S'|, |T_1 \cap T'| \leq 10\sqrt{\eta}n$.*

Proof. Suppose to the contrary that $|S_1 \cap S'|, |T_1 \cap T'| \geq 10\sqrt{\eta}n$. For every $u \in S_1$, the number of vertices in T_1 which are not blue neighbours of u is at most $4\eta n$. If $S' \cap S_1$ and $T' \cap T_1$ both have size at least $10\sqrt{\eta}n$, we deduce $e(G_B[S', T']) \geq 90\eta n$, a contradiction. \square

By the above claim, without loss of generality, we may assume that $|S_2 \cap S'|, |T_1 \cap T'| \leq 10\sqrt{\eta}n$, so $G_B[S_1, T_2]$ is almost empty, and $G_R[S_1, T_2]$ is almost complete. Similar arguments imply that we may assume that $G_B[S_2, T_1]$ is almost complete.

We deduce that there exists $\rho = \rho(\varepsilon)$ such that we may find disjoint sets A_1, A_2, A_3, A_4 satisfying the following conditions.

- $|A_1|, |A_2|, |A_3|, |A_4| \geq (1/4 - \rho)n$.
- $\delta(G_B[A_1, A_2]), \delta(G_B[A_2, A_3]), \delta(G_B[A_3, A_4]) \geq (1/4 - \rho)n$.
- $\delta(G_R[A_1, A_3]), \delta(G_R[A_1, A_4]), \delta(G_R[A_2, A_4]) \geq (1/4 - \rho)n$.

For the time being, we assume that n is even. We obtain a partition $\{A'_1, A'_2, A'_3, A'_4\}$ of $V(G)$ by adding each vertex $u \in V(G) \setminus (A_1 \cup A_2 \cup A_3 \cup A_4)$, to one of the sets A_1, A_2, A_3, A_4 as follows.

$$\text{If } \begin{array}{l} \deg_B(u, A_2) \geq n/32 \\ \deg_B(u, A_3) \geq n/32 \\ \deg_R(u, A_2) \geq n/32 \\ \deg_R(u, A_3) \geq n/32 \end{array} \quad \text{add } u \text{ to } \begin{array}{l} A_3 \\ A_2 \\ A_4 \\ A_1 \end{array}$$

Note that every vertex will be added to one of the A_i 's.

Denote $m_i = |A_i|$. We will find values k_1, k_2, k_3 and l_1, l_2, l_3 such that there exist a partition of $V(G)$ into a blue cycle C_1 and a red cycle C_2 with the following properties.

$$|V(C_1) \cap A_i| = \begin{cases} k_1 & i = 1 \\ k_1 + k_2 & i = 2 \\ k_2 + k_3 & i = 3 \\ k_3 & i = 4 \end{cases} \quad |V(C_2) \cap A_i| = \begin{cases} l_1 + l_2 & i = 1 \\ l_3 & i = 2 \\ l_1 & i = 3 \\ l_2 + l_3 & i = 4 \end{cases}$$

To that end, we find blue paths P_1, P_2, P_3, P_4 forming a blue cycle $C_1 = P_1 P_2 P_3 P_4$ such that the following assertions hold.

- $P_1 \in G_B[A'_1, A'_2]$, its ends are in A'_2 and it has k_1 vertices in A'_1 .
- $P_2, P_4 \in G_B[A'_2, A'_3]$, both have one end in A'_2 and the other in A'_3 and together they have $k_2 + 1$ vertices in A'_2 .
- $P_3 \in G_B[A'_3, A'_4]$, its ends are in A'_3 and it has k_3 vertices in A'_4 .

Similarly, we find red paths Q_1, Q_2, Q_3, Q_4 forming a red cycle $C_2 = Q_1 Q_2 Q_3 Q_4$ such that C_1 and C_2 partition $V(G)$ and the following assertions hold.

- $Q_1 \in G_R[A'_1, A'_3]$, its ends are in A'_1 and it has l_1 vertices in A'_3 .
- $Q_2, Q_3 \in G_R[A'_1, A'_4]$, both have one end in A'_1 and the other in A'_4 and together they have $l_2 + 1$ vertices in A'_1 .
- $Q_4 \in G_R[A'_2, A'_4]$, its ends are in A'_2 and it has l_3 vertices in A'_2 .

The values k_1, k_2, k_3 and l_1, l_2, l_3 clearly need to satisfy the following system of equations.

$$\begin{aligned} m_1 &= k_1 + l_1 + l_2 \\ m_2 &= k_1 + k_2 + l_3 \\ m_3 &= k_2 + k_3 + l_1 \\ m_4 &= k_3 + l_2 + l_3 \end{aligned}$$

Which may be solved as follows.

$$\begin{aligned} k_1 &= k_3 + \frac{1}{2}(m_1 + m_2 - m_3 - m_4) \\ l_1 &= l_3 + \frac{1}{2}(m_1 - m_2 + m_3 - m_4) \\ k_2 &= -(k_3 + l_3) + m_4 \\ l_2 &= -(k_3 + l_3) + m_3 - \frac{1}{2}(m_1 - m_2 + m_3 - m_4). \end{aligned}$$

Since n is even, if k_3, l_3 are integers then so are k_1, k_2, l_1, l_2 . Note that the following inequalities hold.

$$\begin{aligned} k_3 - 2\rho n &\leq k_1 \leq k_3 + 2\rho n & k_2 &\geq (1/4 - \rho)n - (k_3 + l_3) \\ l_3 - 2\rho n &\leq l_1 \leq l_3 + 2\rho n & l_2 &\geq (1/4 - 3\rho)n - (k_3 + l_3). \end{aligned}$$

We pick $l_3 = k_3 = 12\rho n$. It follows that $10\rho n \leq l_1, k_1 \leq 14\rho n$. We now choose the paths P_1, P_2, P_3 as follows. Let P_2 be any edge $u_2 u_3 \in G_B[A_2, A_3]$. Greedily pick paths P_1 in $G_B[A_1, A_2]$ starting with u_2 and ending in some $u_1 \in A_2$, such that P_1 contains k_1 vertices from A_1 . Similarly, let P_3 be a path in $G_B[S_3, A_4]$ with k_3 vertices from A_4 and ends u_3 and u_4 where u_4 is some vertex in A_4 .

We now construct Q_1, Q_2, Q_3 as follows (we implicitly ensure that these paths are disjoint of the paths P_1, P_2, P_3). Let Q_2 be any edge $v_2 v_3 \in G_R[A_1, A_4]$. Pick Q_1 to be a path in $G_R[A'_1, A_3]$ containing all vertices $A'_1 \setminus A_1$, with ends v_2 and $v_1 \in A_1$, and with l_1 vertices from A'_3 . We remark that such a path exists. Indeed, we have $|A'_1 \setminus A_1| \leq 5\rho n$ and any two vertices in $A'_1 \setminus A_1$ may be connected using at most three additional vertices from $A_1 \cup A_3$. Thus we may find a path with at most $10\rho n$ vertices from A_1 starting with v_2 and containing the vertices $A'_1 \setminus A_1$. We extend it arbitrarily to the desired length. Similarly, pick a path $Q_3 \in G_R[A_2, A'_4]$ containing the vertices $A'_4 \setminus A_4$, with ends v_3 and $v_4 \in A_4$ and with l_3 vertices in A_2 .

Denote by U the set of inner vertices in the paths $P_1 P_2 P_3$ and $Q_1 Q_2 Q_3$ and denote $A''_i = A'_i \setminus U$ for $i \in [4]$. By the definition of the paths P_i and Q_i , $i \in [3]$ we obtain the following equalities.

$$\begin{aligned} |A''_1| &= m_1 - k_1 - l_1 = l_2 \\ |A''_2| &= m_2 - k_1 - l_3 = k_2 \\ |A''_3| &= m_3 - k_3 - l_1 = k_2 \\ |A''_4| &= m_4 - k_3 - l_3 = l_2. \end{aligned}$$

It is easy to verify that $G_B[A''_2, A''_3]$ contains a Hamilton path P_4 with ends u_1, u_4 and that $G_R[A''_1, A''_4]$ contains a Hamilton path Q_4 with ends v_1, v_4 .

It remains to consider the case where n is odd. If there exists a vertex u which has blue neighbours

$v_1 \in A_1 \cup A_3$ and $v_2 \in A_2 \cup A_4$, we consider the graph $G \setminus \{u\}$ and partition it into a red cycle and a blue path with ends v_1, v_2 . This may be done by the same arguments as when n is even. Thus we assume that no vertex has blue neighbour in both $A_1 \cup A_3$ and $A_2 \cup A_4$. By symmetry, we may also assume that no vertex has red neighbours in both $A_1 \cup A_2$ and $A_3 \cup A_4$. It follows that every vertex has neighbours in at most three of the sets A_1, A_2, A_3, A_4 , so it sends at least $(1/4 - 2\rho)n$ edges to each of these three sets. We extend the sets A_1, A_2, A_3, A_4 as follows. For each $u \in V(G) \setminus A_1 \cup A_2 \cup A_3 \cup A_4$,

$$\text{if } \begin{array}{l} \deg_B(u, A_2), \deg_R(u, A_3) \geq (1/4 - 2\rho)n \\ \deg_B(u, A_1), \deg_R(u, A_4) \geq (1/4 - 2\rho)n \\ \deg_B(u, A_4), \deg_R(u, A_1) \geq (1/4 - 2\rho)n \\ \deg_B(u, A_3), \deg_R(u, A_2) \geq (1/4 - 2\rho)n \end{array} \quad \text{add } u \text{ to } \begin{array}{l} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}.$$

It is easy to verify that the following claim holds.

Claim 11.3. *The sets A'_1, A'_2, A'_3, A'_4 partition $V(G)$ and satisfy the following conditions.*

$$\begin{aligned} \delta(G_B[A'_1, A'_2]), \delta(G_B[A'_2, A'_3]), \delta(G_B[A'_3, A'_4]) &\geq (1/4 - 2\rho)n. \\ \delta(G_R[A'_1, A'_3]), \delta(G_R[A'_1, A'_4]), \delta(G_R[A'_2, A'_4]) &\geq (1/4 - 2\rho)n. \end{aligned}$$

Without loss of generality, suppose that $|A'_1| \geq n/4$. Then in fact, $|A'_1| > n/4$, because n is odd. Hence $G[A'_1]$ contains an edge uv . If it is blue, u has blue neighbours in both A'_1 and A'_2 . If it is red, u contains red neighbours in both A'_1 and A'_4 . We may continue as before, to partition $G \setminus \{u\}$ into a monochromatic cycle and a monochromatic path. □

The proof of Lemma 11.1 completes the proof of our main Theorem under the assumption that Case (3) from Lemma 8.1 holds.

12 Two monochromatic robust components of size almost $3n/4$

In this section, we consider Case (4) from Lemma 8.1. Similarly to Section 9, we may assume that the given robust components cover $V(G)$.

Lemma 12.1. *Let $\frac{1}{n} \ll \varepsilon, \alpha, \frac{1}{k} \ll 1$ and let G be a graph of order n with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that F_1, F_2 satisfy the following assertions.*

- $F_1 \subseteq G_B$ is (α, k) strongly robust and $|F_1| \geq (3/4 - \varepsilon)n$.
- $F_2 \subseteq G_R$ is (α, k) -robust and $|F_2| \geq (3/4 - \varepsilon)n$.
- $V(G) = V(F_1) \cup V(F_2)$.

Then $V(G)$ may be partitioned into a blue cycle and a red cycle.

We proceed as before, building absorbing paths, and considering the reduced graph on the remaining vertices, where we have a blue component and a red one, each with almost $3/4$ of the vertices. If a perfect matching can be found using the edges in these component, we continue as before to obtain the required partition into cycles. If no such perfect matching exists, we conclude that the graph G satisfies some structural conditions which enable us to either find the required partition “by hand”, or to join two components in a similar way to previous cases.

Proof of Lemma 12.1. Let Q_1, Q_2 be disjoint $\rho^2 n$ -absorbing paths in F_1, F_2 respectively of length at most ρn . As pointed out in Section 10 we may assume that F_2 is weakly robust. Denote its bipartition by $\{X, Y\}$.

Without loss of generality, suppose that $|X \cap V(F_1)| \geq n/8$ and $|Y| \geq \alpha n$. Fix sets $A_1 \subseteq X \cap V(F_1)$ of order $\rho^2 n/2$ and $A_2 \subseteq Y$ of order $\rho^2 n/4$. Finally, apply Lemma 6.1 to the graph G with the vertices in Q_1, Q_2, A_1, A_2 removed, with the cover $\{V(F_1), V(F_2)\}$ and with small enough η . Consider the $(\eta, 6\eta)$ -reduced graph Γ .

Note that, assuming ρ, η are small enough, we have $\delta(\Gamma) \geq (3/4 - \varepsilon)m$, where $m = |\Gamma|$. Furthermore, there is a blue component Φ_1 and a red component Φ_2 which cover $V(\Gamma)$ and are of order at least $(3/4 - 2\varepsilon)m$ each. We consider the subgraph Γ' of Γ spanned by the blue edges in Φ_1 and the red ones in Φ_2 .

Consider the following claim.

Claim 12.2. *One of the following conditions holds.*

1. Γ' has a perfect matching.
2. The following holds for some η, β, l depending only on ε, α, k . There exist subsets $V_1 \subseteq V(F_1) \setminus V(F_2)$, $V_2 \subseteq V(F_2) \setminus V(F_1)$ and $V_0 \subseteq V(F_1) \cap V(F_2)$ such that
 - $|V_1|, |V_2| \geq (1/4 - \eta)n$ and the graphs $G_R[V_1], G_B[V_2]$ are (β, l) strongly robust.
 - $|V_0| \geq (1/2 - \eta)n$ and $G_B[V_1 \cup V_0]$ and $G_R[V_2 \cup V_0]$ are (β, l) -robust, with at least one of them being strongly-robust.

Proof. Let $U_1 = V(\Phi_1) \setminus V(\Phi_2)$, $U_2 = V(\Phi_2) \setminus V(\Phi_1)$ and $U_0 = V(\Phi_1 \cap \Phi_2)$. Note that the vertices in $U_1 \cup U_2$ have degree at least $(1/2 - 4\varepsilon)m$ in Γ' , whereas $|U_0| \geq (1/2 - 4\varepsilon)m$ and the vertices in U_3 have degree at least $(3/4 - 2\varepsilon)m$ in Γ' . If Γ' has no perfect matching, it follows from Theorem 7.1 that $|U_1 \cup U_2| \geq m/2$ and that the set W of vertices $U_1 \cup U_2$ with degree at most $(1/2 + \varepsilon)m$ has size at least $(1/2 - 4\varepsilon)m$. Denote $W_1 = W \cap U_1$ and $W_2 = W \cap U_2$.

We claim that $|W_1|, |W_2| \geq (1/4 - 13\varepsilon)n$. Indeed, the vertices in W_1 have blue degree at most $(1/2 + \varepsilon)m$, so they have red degree at least $(1/4 - 5\varepsilon)m$. Note that the red neighbourhood of W_1 is contained in U_1 . It follows that $|U_1| \geq (1/4 - 5\varepsilon)n$ and similarly $|U_2| \geq (1/4 - 5\varepsilon)n$. Since $|U_0| \geq (1/2 - 4\varepsilon)m$, we have that $|U_1|, |U_2| \leq (1/4 + 9\varepsilon)m$. Thus $|W_1|, |W_2| \geq (1/4 - 13\varepsilon)m$. It is easy to verify that $\Gamma_R[W_1]$ and $\Gamma_B[W_2]$ are connected and non bipartite (in fact, they are almost complete).

Let W_0 be the intersection of the blue neighbourhood of W_1 in U_0 and the red neighbourhood of W_2 in U_0 . By the definition of the sets U_i , there are no edges between U_1 and U_2 , there are no red edges between U_1 and U_0 and no blue edges between U_2 and U_0 . It follows that the each

vertex in U_1 has at least $(1/2 - 10\varepsilon)m$ blue neighbours in U_0 , and by the analogous argument for U_2 , we have $|W_0| \geq (1/2 - 20\varepsilon)m$. Note that $\Gamma_B[W_0 \cup W_2]$ and $\Gamma_R[W_0 \cup W_2]$ are connected and it is not hard to see that at least one of them is non-bipartite. It follows from Lemma 6.7 that condition (2) holds. \square

Similarly to the proofs in Section 10, one of the following holds.

1. There are two vertex-disjoint blue paths between V_2 and $V_1 \cup V_0$.
2. There are two vertex-disjoint red paths between V_1 and $V_2 \cup V_0$.
3. There exists a set X of size at most 2 such that the sets V_2 and $V_1 \cup V_0$ are not connected in $G_B \setminus X$ and the sets V_1 and $V_2 \cup V_0$ are not connected in $G_R \setminus X$.

In the first two cases we proceed as in Section 10 to join say the two blue component to form an almost spanning component, which together with the large red component ($V_2 \cup V_0$) may be used to find the required cycle partition. Thus we assume that the Case (3) holds.

Consider the graph $G' = G \setminus X$. Let U_1 be the component of V_1 in G'_R and let U_2 be the blue component of V_2 in G'_B . Then $e(G[U_1, U_2]) = 0$, so $|U_1|, |U_2| \leq n/4 - 1$.

Denote $W = V(G') \setminus (U_1 \cup U_2)$ (so $|W| \geq n/2$). We define U'_1, U'_2 as follows. For each $x \in X$, if $\deg_B(x, W) \geq n/8$, put x into U_1 . Otherwise we have $\deg_R(x, W) \geq n/8$, and we put x into U_2 .

Note that $e(G[W]) \geq n^2/16$. We may assume without loss of generality that $e(G_B[W]) \geq n^2/32$. Denote $|W| = |U'_1| + |U'_2| + k$ and note that $0 \leq k \leq 5\eta n$.

Claim 12.3. *There exists $\theta = \theta(\varepsilon)$ such that W may be partitioned into sets W_1, W_2 satisfying the following conditions.*

- $|W_1| = |U'_1| + k$ and $|W_2| = |U'_2|$.
- $G_B[W_1]$ contains a blue path of length k .
- The graphs $G_B[U'_1, W_1]$ and $G_R[U'_2, W_2]$ have minimum degree at least $n/32$ and all but at most θn vertices have degree at least $(1/4 - \theta)n$.

It is easy to conclude from Claim 12.3 that G may be partitioned into a blue cycle and a red one. Indeed, the graph $G_R[U'_2, W_2]$ is Hamiltonian (e.g. by Corollary 7.2). We claim that $G_B[U'_1 \cup W_1]$ is also Hamiltonian, completing the partition of $V(G)$ into a red cycle and a blue one. Indeed, take any path P in $G_B[W_1]$ of length k . Denote its ends by u, v and let W'_1 be the set obtained from W_1 by removing the inner vertices of P . It is easy to see (e.g. by Corollary 7.2) that $G_B[U'_1, W'_1]$ has a Hamilton path with ends u, v . It remains to prove Claim 12.3.

Proof of Claim 12.3. Recall that $(1/4 - \eta)n \leq |U_1|, |U_2| \leq n/4 - 1$. Denote by W' the set of vertices which have at least $\sqrt{\eta}n$ non-neighbours in either U_1 or U_2 . Note that each vertex in U_1 has at most ηn non-neighbours in W . It follows that $|W'| \leq \sqrt{\eta}n$.

Let W'_1 be the set of vertices in W' with (blue) degree at least $n/16$ into U'_1 , and let $W'_2 = W' \setminus W'_1$. So the vertices in W'_2 have degree at least $n/16$ into U'_2 .

Note that $e(G_B[W \setminus W']) \geq n^2/64$. It follows from Theorem 7.3 that $W \setminus W'$ contains a blue path P of length k .

Pick any partition $\{W_1, W_2\}$ with the following properties.

- $|W_1| = |U'_1| + k$ and $|W_2| = |U'_2|$.
- $W'_1 \cup V(P) \subseteq W_1$ and $W'_2 \subseteq W_2$.

Such a partition satisfies the required conditions of Claim 12.3 with $\theta = \sqrt{\eta}$. □

□

13 Four half-sized robust components

In this section we consider Case (5) from Lemma 8.1.

Lemma 13.1. *Let $\frac{1}{n} \ll \alpha, \frac{1}{k} \ll \varepsilon \ll 1$ and let G be a graph of order n with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that F_1, F_2, F_3, F_4 satisfy the following conditions.*

- $F_1, F_2 \subseteq G_B$ are vertex-disjoint (α, k) -robust components on at least $(1/2 - \varepsilon)n$ vertices.
- $F_3, F_4 \subseteq G_R$ are vertex-disjoint (α, k) -strongly robust components on at least $(1/2 - \varepsilon)n$ vertices.
- $V(F_1) \cup V(F_2) = V(F_3) \cup V(F_4)$.
- $|V(F_i) \cap V(F_j)| \geq (1/4 - \varepsilon)n$ for $i \in [2], j \in [3, 4]$.

Then $V(G)$ may be partitioned into a blue cycle and a red one.

We follow similar ideas to previous sections. If there exist four vertex-disjoint paths, two of which are blue and connect F_1 with F_2 and two are red and connect F_3 with F_4 , then we may continue as in previous sections, by essentially having two robust components, one red and one blue, which both span almost all the vertices. The main effort in this case goes into showing that if such paths do not exist, the desired partition may be found by Lemma 7.14.

Proof of 13.1. We extend the components F_i as follows. For every vertex v not in any of the components, if there exist $i \in \{1, 2\}, j \in \{3, 4\}$ such that v sends at least αn blue edges to F_i and at least αn red edges to F_j , we add v to F_i and F_j . Note that if no such i, j exist, then v either blue degree at least $(3/4 - 3\varepsilon)n$ or red degree at least $(3/4 - 3\varepsilon)n$.

Note that the obtained components satisfy the conditions above (though with relaxed parameters α, k in the definition of robustness and with say 2ε instead of ε). We abuse notation by denoting the modified components by F_1, F_2, F_3, F_4 . So in addition to the above conditions, we have that every vertex not in $V(F_1) \cup V(F_2)$ has either blue degree or red degree at least $(3/4 - 3\varepsilon)n$.

We claim that one of the following assertions holds.

1. There exist vertex-disjoint paths P_1, P_2, P_3, P_4 such that P_1, P_2 are blue paths from F_1 to F_2 and P_3, P_4 are red paths from F_3 to F_4 .
2. There exist two vertices u, v such that F_1, F_2 belong to different connected components of $G_B \setminus \{u, v\}$. Furthermore, v sends at most εn blue edges to either F_1 or F_2 .
3. There exist two vertices u, v such that F_3, F_4 belong to different connected components of $G_R \setminus \{u, v\}$. Furthermore, v sends at most εn red edges to either F_3 or F_4 .

Condition (1) implies that we may connect F_1 and F_2 using P_1, P_2 to obtain a large robust blue component, and similarly we may connect F_3 and F_4 using the paths P_3, P_4 to obtain a large strongly robust red component. We may continue as in Section 10 to obtain the desired partition of $V(G)$ into a red cycle and a blue one.

If one of Conditions (2, 3) holds, we may find the desired partition into a red cycle and a blue one by Lemma 7.14.

It remains to prove that indeed, one of the above three cases holds. We call a vertex *blue* if it sends at least αn blue edges to both F_1 and F_2 . Similarly, a vertex is *red* if it sends at least αn red edges to both F_3 and F_4 . We show that either one of the above three conditions holds, or there are at least four vertices which are either blue or red.

If true, we conclude that one of the three conditions above holds. It is easy to verify that if there exist four vertices, two of them red and two blue, we may find paths as Condition (1). Thus we may assume that there is at most one red vertex. This implies that either Condition (3) holds, or there are three vertex disjoint red paths P_1, P_2, P_3 between F_3 and F_4 . We may assume that the inner vertices of these path are not in $V(F_3) \cup V(F_4)$. If there exist four blue vertices u_1, u_2, u_3, u_4 , without loss of generality, $u_1, u_2 \notin V(P_1) \cup V(P_2)$ and we may find two vertex-disjoint blue paths between F_1 and F_2 which are disjoint of P_1, P_2 using the blue vertices u_1, u_2 (we can take them to be of length 2 and be centred at u_1 and u_2). If there exist three blue vertices u_1, u_2, u_3 and a different red vertex v , without loss of generality, the path P_1 has length 2, is centred at v and avoids u_1, u_2, u_3 . We may further assume that P_2 does not contain u_1, u_2 . It follows that we may find two vertex-disjoint blue paths, disjoint of P_1, P_2 , between F_1 and F_2 .

It remains to show that indeed, if Conditions (1 - 3) do not hold, there are at least four vertices which are either red or blue. Clearly, each vertex $v \in V(G) \setminus (F_1 \cup F_2)$ is either red or blue. Let l be the number of these vertices. Denote

$$\begin{aligned} A_1 &= V(F_1) \cap V(F_3) \\ A_2 &= V(F_1) \cap V(F_4) \\ A_3 &= V(F_2) \cap V(F_3) \\ A_4 &= V(F_2) \cap V(F_4). \end{aligned}$$

Consider the following claim.

Claim 13.2. *Suppose that $|A_i| \geq n/4 - 1 + k$ where $1 \leq k \leq 4$. Then either one of the Conditions (1 - 3) from above holds or A_i contains at least k vertices which are either blue or red.*

Using the claim, if the above conditions do not hold and there are at most three vertices which are either blue or red, we have $l \leq 3$ and the total number of vertices is at most $4(n/4 - 1) + (3 - l) + l \leq n - 1$, a contradiction. It remains to prove the claim.

Proof of Claim 13.2. Without loss of generality, $i = 1$. Note that if $G[A_1, A_4]$ has a matching of size 7, Condition (1) holds. Indeed, let M be such a matching. If in such a matching at least two edges are red and at least two are blue, condition (1) holds, since each edge connects F_1 and F_2 as well as F_3 and F_4 . Thus we may assume that at most one edge is red.

Recall that there exist vertex-disjoint red paths P_1, P_2 between F_3 and F_4 (otherwise Condition (3) holds). We assume that the inner vertices of P_1 and P_2 do not belong to $V(F_3 \cup F_4) = V(F_1 \cup F_2)$, so each of the paths P_1 and P_2 intersects at most two edges of M . It follows that there exist two blue edges $e_1, e_2 \in M$ which are disjoint of P_1 and P_2 , and Condition (1) holds. It remains to consider the case where $G[A_1, A_4]$ has no matching of size 7.

We deduce that there is a set $U \subseteq A_1 \cup A_4$ of at most six vertices which intersects each of the edges of $G[A_1, A_4]$. Note that by the minimum degree conditions, every vertex in A_4 has at least k neighbours in A_1 . Hence, $|U \cap A_1| \geq k$. Denote $A'_4 = A_4 \setminus U$. Then

$$e(G[A_1, A'_4]) \geq k(|A_4| - 6).$$

We conclude that at least k vertices in $U \cap A_1$ have at least $n/25$ neighbours in A_4 . Indeed, otherwise we have

$$e(G[A_1, A'_4]) \leq (k-1)|A_4| + 6n/25 < k(|A_4| - 6),$$

a contradiction.

Each of the vertices in A_1 with at least $n/25$ neighbours in A_4 is either red or blue. E.g. if it sends at least αn blue edges to A_4 is blue. \square

\square

14 Large strongly robust blue component

In this section, we resolve Case (1) from Lemma 8.1, which is the last remaining case.

Lemma 14.1. *Let $\frac{1}{n} \ll \varepsilon, \alpha, \frac{1}{k} \ll 1$ and let G be a graph of order n with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that F is a blue (α, k) -strongly robust component on at least $(1 - \varepsilon)n$ vertices. Then $V(G)$ may be partitioned into a blue cycle and a red cycle.*

In our proof of Lemma 14.1, we extend F to include all vertices which send a fairly large number of blue edges into F , and denote the set of remaining vertices by Z . We consider two cases, according to the size of Z .

The case where Z is relatively small turns out to be harder. We apply the Regularity Lemma on F and prove a structural result on the reduced graph, focusing on ways to obtain perfect matchings. In each of the cases for the structure of the reduced graph, we can partition almost all of the vertices into a red cycle and a blue one. The vertices of Z may be covered using their large degree and the leftover vertices of F may be absorbed as usual.

In the case where Z is large, the reduced graph can easily be seen to have a perfect matching consisting of a connected blue matching and a connected red matching. We have to be slightly

more careful than usual when obtaining cycles from the connected matching so as to cover Z . The leftover vertices of F can be absorbed as usual.

Proof of Lemma 14.1. Let F_1 be the graph obtained by adding to F the vertices in G with at least αn blue neighbours in F . By Lemma 5.4, F_1 is $(\alpha^3/2, k+2)$ strongly robust. Denote $Z = V(G) \setminus V(F_1)$. Note that

$$\deg_R(z) \geq (3/4 - \varepsilon - \alpha)n \geq (3/4 - 2\varepsilon)n \text{ for every } z \in Z.$$

It follows that every two vertices in Z have at least $(1/2 - 4\varepsilon)n$ red neighbours in common. We consider two cases according to the size of Z . Pick $\beta > 0$ by Lemma 5.2 so that F_1 remains $(\alpha^3/4, k+2)$ strongly robust after removing at most βn vertices.

Case 1: $|Z| \geq \beta n$

Let Q be a $\rho^2 n$ -absorbing path in F_1 of length at most ρn . Pick a suitably small η , and let G' be the subgraph obtained by applying Lemma 6.1 with the graph $G \setminus V(Q)$, the partition $\{V(F_1), Z\}$ and parameters η and $d = 6\eta$. Let Γ be the corresponding reduced graph. As usual, $\delta(\Gamma) \geq (3/4 - 2\varepsilon)m$, where $m = |\Gamma|$.

Let Φ_1 be the blue subgraph spanned by the clusters which are contained in $V(F_1)$. Then $|\Phi_1| \geq (1 - 2\varepsilon)m$ and Φ_1 is connected by Lemma 6.8. The vertices in clusters contained in Z have red degree at least $(3/4 - 4\varepsilon)m$ in Γ (note that since $|Z| \geq \beta n \geq 2\eta n$, there are such clusters). In particular, Γ contains a red component Φ_2 of order at least $(3/4 - 4\varepsilon)m$. It is easy to check that $\Phi_1 \cup \Phi_2$ has a perfect matching, e.g. by Theorem 7.1.

Let U_1 be the set of vertices in G' belonging to clusters of the blue matching and let U_2 be the set of vertices in G' belonging to clusters of the red matching. We obtain the required partition into a blue cycle and a red cycle as follows.

Suppose first that $|U_2| \leq 3n/8$. Fix two vertices $z_1, z_2 \in Z$. Let W be a set of vertices in G' containing 3η of the vertices of each cluster in $V(F_1)$. Clearly, z_1, z_2 each have many neighbours in the set of vertices belonging to the clusters of Φ_2 . Thus, by Lemma 6.9, there is a red path P_1 in G' between z_1 and z_2 spanning at least $(1 - 6\eta)$ of the vertices of $U_2 \setminus W$ and using at most m^2 other vertices.

Denote by Z' the set of vertices in Z which are not covered by this path. Note that $|Z'| \leq 9\eta n$. Furthermore, every two vertices of $Z' \cup \{z_1, z_2\}$ have at least $n/16$ common red neighbours in $V(G') \setminus U_2$. It is thus possible to find a path in $G' \setminus (V(P_1) \cup U_1)$ between z_1 and z_2 , containing Z' and using at most 200η of the vertices of each cluster. Indeed, such a path can be constructed greedily. Suppose $Z' = \{z_3, \dots, z_t\}$. For $i = 2, \dots, t$ we find a common red neighbour of z_i, z_{i+1} (where subscripts are taken modulo t) which was not used before and which does not belong to a cluster in U_2 with at least 200η of its vertices already used. Note that this would give the required red path, completing P_1 to a red cycle C . In each step, there are at least $n/16$ possible choices, out of which at most $9\eta n$ were already used, and at most $9\eta n/200\eta < n/20$ belong to clusters for which at least 200η of its vertices are used. Thus it is possible to choose a suitable vertex.

We now construct a blue cycle which is disjoint of C , contains Q and misses at most 210η of the vertices of each cluster in U_1 , using Lemma 6.9 (we use the vertices of W to connect cluster pairs, as described after the statement of Lemma 6.9). The missing vertices may be absorbed by Q , completing the desired cycle partition.

Suppose now that $|U_2| \geq 3n/8$. By Lemma 6.9, there exist a blue cycle C_1 and a red cycle C_2 which are disjoint and C_i covers all but at most 9η of the vertices of U_i for $i \in [2]$. In particular, the red cycle has length at least $5n/16$. Let Z' be the set of vertices of Z which are not covered by either of the cycles. We show how to obtain a red cycle C'_2 such that $Z \subseteq V(C'_2) \subseteq V(C_2) \cup Z$ and $|V(C'_2) \setminus V(C_2)| \leq 60\eta n$. To that end, we claim that the vertices of Z' can be inserted one by one, such that in each stage at most 20 of the original vertices of C are removed, none of them from Z . If z cannot be inserted as explained, the number of red neighbours of z in the cycle is at most $40|Z| + n/20$. But every vertex $z \in Z'$ has at least $(1/16 - 3\epsilon)n$ red neighbours in the cycle obtained from C_2 , as long as it has length at least $(5/16 - 60\eta)n$, implying that z may indeed be inserted. There are at most $20|Z'| + 9\eta n \leq \rho^2 n$ vertices missing from $V(C_1) \cup V(C'_2)$, all of them from $V(F_1)$. They can be absorbed by Q .

Case 2: $|Z| \leq \beta n$

If $|Z| \geq 3$, fix three vertices $z_1, z_2, z_3 \in Z$. Let P be a red path containing $Z \setminus \{z_1, z_2, z_3\}$ (but avoiding z_1, z_2, z_3) with at most βn vertices of F_1 (note that P may be constructed greedily). If $|Z| \leq 3$, we take P to be empty. Let $F_2 = F_1 \setminus V(P)$. F_2 is $(\alpha/4, k+2)$ strongly robust by the choice of β . Apply Lemma 5.5 to obtain a $\rho^2 n$ -absorbing path Q in F_2 of length at most ρn , and denote $F_3 = F_2 \setminus V(Q)$. F_3 is $(\alpha^3/8, k+2)$ strongly robust, by Lemma 5.2. Furthermore, $|F_3| \geq (1 - 2\epsilon)n$ and the vertices of Z have at least $(3/4 - 4\epsilon)n$ red neighbours in F_3 .

Apply the Regularity Lemma, Lemma 6.1, to the graph $G[V(F_3)]$ with a suitably small parameter η . Let Γ be the corresponding $(\eta, 4\eta)$ -reduced graph. We have $\delta(\Gamma) \geq (3/4 - 2\epsilon)m$ and by Lemma 6.8, Γ_B is connected.

We shall use the following proposition.

Proposition 14.2. *One of the following assertions holds.*

1. Γ_B has a perfect matching.
2. There exists a red component Φ on at least $(1/2 - 50\epsilon)m$ vertices such that $\Gamma_B \cup \Phi$ has a perfect matching.
3. There exist disjoint subsets $X_1, X_2 \subseteq V(G)$ of size at least $(1/4 - 500\epsilon)m$ such that
 - $X_1 \cup X_2$ is independent in Γ_B .
 - $\Gamma_R[X_i]$ is connected and $\Gamma_B \cup \Gamma_R[X_i]$ has a perfect matching for $i \in [2]$.
4. There exist disjoint subsets X_1, X_2, Y_1, Y_2 of size at least $(1/4 - 500\epsilon)m$ such that
 - $\Gamma_R[X_i \cup Y_i]$ is connected and non-bipartite for $i \in [2]$.
 - $\Gamma_B[X_1 \cup X_2]$ and $\Gamma_B[Y_1 \cup Y_2]$ are connected.

Before proving Proposition 14.2, we show how to complete the proof of Lemma 14.1 in this case using the proposition.

Γ_B has a perfect matching

Suppose that Γ_B has a perfect matching. Recall that Γ_B is connected, so this matching is connected. We complete $Z \cup V(P)$ to a red cycle C_1 using at most four additional vertices of F_3 . By Lemma 6.9, there exists a blue cycle C_2 , disjoint of C_1 which extends the absorbing path Q and contains all but at most $6\eta n \leq \rho^2 n$ vertices of F_3 . The remaining vertices can be absorbed by Q to obtain a blue cycle C'_2 . The cycles C_1, C'_2 form the required cycle partition.

An almost half-sized red component whose union with Γ_B has a perfect matching

Suppose that Φ is a red component of size at least $(1/2 - 50\varepsilon)m$ and $\Gamma_B \cup \Phi$ has a perfect matching. Define a red path P' as follows. If $|Z| \geq 2$, extend P to a path P' containing Z with ends $z_1, z_2 \in Z$ (we may do this using at most three additional vertices). If $|Z| = 1$, take P' to be the path (z) where $Z = \{z\}$. If $Z = \emptyset$, take P' to be the empty path. Note that every vertex of Z sends many red edges to the set of vertices contained in the clusters defined by $V(\Phi)$. By Lemma 6.9, there exist vertex-disjoint cycles C_1, C_2 such that C_1 is blue and contains the absorbing path Q and C_2 is red and contains the path P' . Furthermore, the cycles C_1, C_2 cover all but at most $\rho^2 n$ vertices belonging to F_3 , which may be absorbed by Q , completing the desired partition into a red cycle and a blue one.

Two almost quarter-sized red components whose union with Γ_B has a perfect matching

Let $X_1, X_2 \subseteq V(\Gamma)$ be disjoint sets of size at least $(1/4 - 500\varepsilon)m$ satisfying the following conditions.

- $X_1 \cup X_2$ is independent in Γ_B .
- $\Gamma_R[X_i]$ is connected and $\Gamma_B \cup \Gamma_R[X_i]$ has a connected matching for $i \in [2]$.

Note that we may assume that $\Gamma_R[X_1 \cup X_2]$ is not connected, since otherwise we may proceed as in the previous case.

Let U_i be the set of vertices contained in the clusters in X_i . Note that $|U_i| \geq (1/4 - 501\varepsilon)n$. We define a path P' as follows. If $|Z| \geq 3$, without loss of generality, z_1, z_2 send at least $2\eta n$ red edges into U_1 . Extend P to a path P' containing Z with ends z_1, z_2 (using at most three additional vertices). If $|Z| = 1$, denote $Z = \{z\}$ and take $P' = (z)$, and suppose without loss of generality that z has at least $2\eta n$ red neighbours in U_1 . If $Z = \emptyset$, take P' to be the empty path. As before, since $\Gamma_B \cup \Gamma_R[X_1]$ has a perfect matching, Lemma 6.9 implies that there exist disjoint cycles C_1, C_2 such that C_1 is blue and contains Q and C_2 is red and contains P' , and together they cover all but at most $\rho^2 n$ vertices of F_3 , which may be absorbed by C_1 .

It remains to consider the case where $|Z| = 2$. Denote $Z = \{z_1, z_2\}$. If for some $i \in [2]$, both z_1, z_2 have at least $2\eta n$ red neighbours in U_i , we may continue as before by taking P' to be a path of length 2 connecting z_1, z_2 . Thus we assume that

$$\deg_R(z_1, U_2), \deg_R(z_2, U_1) \leq 2\eta n \leq \varepsilon n.$$

Recall that $\deg_R(z_i) \geq (3/4 - 4\varepsilon)n$ for $i \in [2]$. It follows that $|U_1|, |U_2| \leq (1/4 + 5\varepsilon)n$. Let W be the set of common red neighbours of z_1 and z_2 in $V(G) \setminus (U_1 \cup U_2)$. Then $|W| \geq (1/2 - 20\varepsilon)n$.

Suppose that there exists a vertex $w_1 \in W$ with at least $4\eta n$ red edges into $U_1 \cup U_2$. Without loss of generality, w_1 has at least $2\eta n$ red neighbours in U_1 . Take $P' = (w_1 z_2 w_2 z_1)$ for some $w_2 \in W$ and continue as before (when $|Z| \neq 2$) to conclude that the desired partition exists.

We now assume that every vertex in W has at most εn red neighbours in $U_1 \cup U_2$. Furthermore, by the definition of the reduced graph, every vertex in $U_1 \cup U_2$ has at most $9\eta n \leq \varepsilon n$ red neighbours in W . Note that since $e(\Gamma[X_1, X_2]) = 0$, the graph $G_B[U_1 \cup U_2, W]$ is almost complete. It follows that we may apply Lemma 7.13 with parameter 1002ε , to conclude that $V(G)$ may be partitioned into a red cycle and a blue one.

Four half-sized monochromatic components

Suppose that X_1, X_2, Y_1, Y_2 are disjoint sets of size at least $(1/4 - 500\varepsilon)m$ with the following properties.

- $\Gamma_R[X_i \cup Y_i]$ is connected and non-bipartite for $i \in [2]$.
- $\Gamma_B[X_1 \cup X_2]$ and $\Gamma_B[Y_1 \cup Y_2]$ are connected.

We conclude from Lemma 6.7 that there exist sets $U_1, U_2, W_1, W_2 \subseteq V(G)$ of order at least $(1/2 - \theta)n$, where $\theta = 502\varepsilon$, such that

- U_1, U_2 are disjoint, W_1, W_2 are disjoint and $U_1 \cup U_2 = W_1 \cup W_2$.
- $|U_i \cap W_j| \geq (1/4 - \theta)n$ for $i, j \in [2]$.
- $G_B[U_i]$ is (γ, l) -robust and $G_R[W_i]$ is (γ, l) -strongly robust for $i \in [2]$, where $\gamma = \gamma(\varepsilon, \alpha, k)$ and $l = l(\varepsilon, \alpha, k)$.

By Lemma 13.1, $V(G)$ may be partitioned into a red cycle and a blue one. This completes the proof of Lemma 14.1, under the assumption that Proposition 14.2 holds. We prove it in the following subsection.

□

Proof of Proposition 14.2

In this subsection, we prove Proposition 14.2. We shall consider four cases according to the sizes of the red components in Γ . In each of these cases, we apply Lemmas 7.9 or 7.7 to gain additional structural information about the graph Γ in case Γ_B has no perfect matching (otherwise we are done). This information will enable us to show that one of the conditions in the proposition holds.

Proof of Proposition 14.2. The following claim reduced the proof to the following four cases.

Claim 14.3. *One of the following conditions holds.*

1. Γ_R has no components of order at least $(1/4 - 4\epsilon)m$.
2. Γ_R contains a component of order at least $(1/2 - 50\epsilon)m$.
3. There exists a tripartition $\{X_1, X_2, X_3\}$ of $V(\Gamma)$ such that $|X_i| \leq (1/2 - 24\epsilon)m$ and every red component is contained in one of the red sets X_i .
4. Γ_R contains four vertex-disjoint components of order at least $(1/4 - 36\epsilon)m$.

Proof. Let $s_1 \geq \dots \geq s_l$ be the orders of components in Γ_R . We may assume that $(1/4 - 4\epsilon)m \leq s_1 \leq (1/2 - 50\epsilon)m$, otherwise one of the first two conditions holds.

If $s_1 + s_2 \geq (1/2 + 24\epsilon)m$, we take the tripartition of $V(\Gamma)$ corresponding to the partition $\{[1], [2], [3, \dots, l]\}$ of $[s]$. Thus we can assume that $s_1 + s_2 \leq (1/2 + 24\epsilon)m$.

If $s_3 \leq 50\epsilon m$, we may find the desired partition as follows. Note that $s_2 \leq (1/4 + 12\epsilon)m$ and pick i to be the minimal such that $s_2 + \dots + s_i > 3m/8$. Since $s_i \leq s_3 \leq 50\epsilon m$, we have $s_2 + \dots + s_i \leq (3/8 + 50\epsilon)m$ and $s_{i+1} + \dots + s_l \leq (3/8 - 54\epsilon)m$. It follows that the partition $\{[1], [2, i], [i+1, l]\}$ satisfies the requirements. Thus, we may assume that $s_3 \geq 50\epsilon m$.

If $s_1 + s_2 \leq (1/2 - 24\epsilon)m$, we have $s_1 + s_2 \geq (1/4 + 46\epsilon)m$. Let i be minimal such that $s_3 + \dots + s_i \geq (1/4 - 12\epsilon)m$. Since $s_i \leq s_2 \leq (1/4 - 12\epsilon)m$, we have $s_3 + \dots + s_i \leq (1/2 - 24\epsilon)m$. The partition $\{[2], [3, i], [i+1, l]\}$ satisfies the requirements. We may now assume that $(1/2 - 24\epsilon)m \leq s_1 + s_2 \leq (1/2 + 24\epsilon)m$.

Since $s_3 \geq 50\epsilon m$, we have $s_1 + s_2 + s_3 \geq (1/2 + 24\epsilon)m$. If $s_2 + s_3 \leq (1/2 - 24\epsilon)m$, the partition $\{[1], [2, 3], [4, l]\}$ satisfies the requirements. Otherwise, $s_2 + s_3 \geq (1/2 - 24\epsilon)m$. It follows that $s_1 \geq s_2 \geq (1/4 - 12\epsilon)m$, hence $s_1 \leq (1/4 + 36\epsilon)m$ and $s_3 \geq (1/4 - 60\epsilon)m$. If $s_3 + s_4 \leq (1/2 - 24\epsilon)m$, we may take the partition $\{[1], [2] \cup [5, l], [3, 4]\}$. Otherwise, we have $s_4 \geq (1/4 - 36\epsilon)m$, and the required four components exist.

□

We prove Proposition 14.2 in each of the four cases in Claim 14.3. The second case, where there is a large red component, turns out to be hardest and we leave it to the end of the proof.

No large red components

In Case (1) of the previous claim, we have $\deg_B(\Gamma) \geq m/2$, implying that Γ_B has a perfect matching.

Tripartition

In Case (3) of Claim 14.3, there exists a tripartition $\{X_1, X_2, X_3\}$ of $V(\Gamma)$ such that $|X_i| \leq (1/2 - 24\epsilon)m$ and every red component is contained in one of the red sets X_i . We assume that Γ_B contains no perfect matching. By Lemma 7.7, without loss of generality, there exist subsets $Y_1 \subseteq X_1, Y_2 \subseteq X_2$ such that $|Y_1|, |Y_2| \geq (1/4 - 10\epsilon)m$ and $Y_1 \cup Y_2$ is independent in Γ_B .

It follows that

$$\delta(\Gamma_R[Y_i]) \geq |Y_i| - 12\varepsilon n. \quad (1)$$

In particular, $\Gamma_R[Y_i]$ is connected. We show that $\Gamma_B \cup \Gamma_R[Y_i]$ has a perfect matching for $i \in [2]$. Suppose to the contrary that $\Phi = \Gamma_B \cup \Gamma_R[Y_1]$ has no perfect matching. By Lemma 7.7, it follows that there exist subsets $Z_i \subseteq X_i$ and $Z_j \subseteq X_j$ of size at least $(1/4 - 10\varepsilon)n$ such that $Z_1 \cup Z_2$ is independent in Φ , for some $1 \leq i \neq j \leq 3$.

Suppose first that say $i = 2$. Then the intersection of Y_2 and Z_2 is non-empty. Let $u \in Y_2 \cup Z_2$. Note that u has at most $(1/4 + 2\varepsilon)m$ non-neighbours in $X_1 \cup X_3$, at least $(1/4 - 10\varepsilon)m$ of which belong to Y_1 , so $|Z_1 \cap Y_1| \geq (1/4 - 22\varepsilon)m$, a contradiction to (1).

It remains to consider the case where $i = 1, j = 3$. As before, we may pick $u \in Y_1 \cap Z_1$. But u has at least $|Z_3| + |Y_2| \geq (1/2 - 20\varepsilon)m$ non-neighbours in Γ , a contradiction.

Almost equipartition into four parts

In Case (4) of 14.3, $\{X_1, X_2, X_3, X_4\}$ is a partition of $V(G)$ such that every red component is contained in one of the parts X_i and $|X_i| \geq (1/4 - 36\varepsilon)m$. The following claim finishes the proof of Proposition 14.2 in this case.

Claim 14.4. *One of the following assertions holds.*

1. Γ_B has a perfect matching.
2. For some $1 \leq i < j \leq 4$, there exist subsets $Y_i \subseteq X_i$ and $Y_j \subseteq X_j$ of size at least $(1/4 - 500\varepsilon)m$ such that
 - $\Gamma_R[Y_l]$ is connected for $l \in \{i, j\}$.
 - $Y_i \cup Y_j$ is independent in Γ_B .
 - $\Gamma_B \cup \Gamma_R[Y_l]$ has a perfect matching for $l \in \{i, j\}$.

Proof. Consider Lemma 7.9 with the graph Γ_B , the bipartition $\{X_1 \cup X_2, X_3 \cup X_4\}$ and parameter 72ε . Assuming that Γ_B has no perfect matching, it is easy to verify that the first three conditions cannot hold. It follows that Γ_B has an independent set Y of size at least $(1/2 - 432\varepsilon)m$. This implies that $\delta(\Gamma_R[Y]) \geq (1/4 - 434\varepsilon)m$, implying that $\Gamma_R[Y]$ is the union of two red connected subgraphs of order at least $(1/4 - 434\varepsilon)m$. Without loss of generality, $Y \subseteq X_1 \cup X_2$. Denote $Y_i = Y \cap X_i$. If $|X_1| + |X_2| \geq |X_3| + |X_4|$, the graph $\Gamma_B \cup \Gamma_R[Y_i]$ has a perfect matching for $i \in [2]$, by Lemma 7.9. Otherwise, $\Gamma_B[X_1 \cup X_2, X_3 \cup X_4]$ is almost complete, and by Lemma 7.9, if Γ_B has no perfect matching, $X_3 \cup X_4$ contains an independent set Y of size at least $(1/2 - 432\varepsilon)m$. Denoting $Y_i = Y \cap X_i$ for $i \in [3, 4]$, we conclude as before that $|Y_i| \geq (1/4 - 434\varepsilon)m$, $\Gamma_R[Y_i]$ is connected and $\Gamma_B \cup \Gamma_R[Y_i]$ has a perfect matching for $i \in [3, 4]$. \square

Large red component

We now consider Case (2) of Claim 14.3, which is the last remaining case. Let Φ_1 be a red component on at least $(1/2 - 50\varepsilon)m$ vertices and denote $X_1 = V(\Phi_1)$, $X_2 = V(\Gamma) \setminus X_1$ and $\Gamma_1 = \Gamma_B \cup \Phi_1$.

The following claim is a simple application of Theorem 7.1.

Claim 14.5. *One of the following assertions holds.*

1. Γ_1 has a perfect matching.
2. X_2 contains a red component of order at least $(1/2 - 4\varepsilon)m$.
3. $|X_2| \geq m/2$, and $\Gamma_R[X_2]$ contains two red components of order at least $(1/4 - 52\varepsilon)m$.

Proof. Suppose that Γ_1 does not have a perfect matching. In particular, Γ_1 has no Hamilton cycle. Note that

$$\deg_{\Gamma_1}(u) \geq \begin{cases} (3/4 - 2\varepsilon)m & u \in X_1 \\ (3/4 - 2\varepsilon)m - (n - |X_1|) & u \in X_2. \end{cases}$$

It follows from Theorem 7.1 that either there are at least $m/2$ vertices of degree at most $(1/2 + 50\varepsilon)m$ or there are at least $(1/4 + 2\varepsilon)m$ vertices of degree at most $(1/4 + 2\varepsilon)m$.

Suppose that the former holds and let Y be the set of vertices of degree at most $(1/2 + 50\varepsilon)m$ in Γ_1 . Then $\deg_R(u, X_2) \geq (1/4 - 52\varepsilon)m$ for every $u \in Y$. It follows that either $\Gamma_R[Y]$ is connected, or it consists of two connected components of order at least $(1/4 - 52\varepsilon)m$.

Suppose now that the latter holds, and let Y be the set of vertices of degree at most $(1/4 + 2\varepsilon)m$ in Γ_1 . Then $\deg_R(u, X_2) \geq (1/2 - 4\varepsilon)m$ for every $u \in Y$, and in particular, X_2 contains a red component of order at least $(1/2 - 4\varepsilon)m$.

□

In order to prove Proposition 14.2, we may assume that Γ_1 has no perfect matching. We consider first the third case of Claim 14.5, where $|X_2| \geq m/2$ and X_2 contains two red components of size at least $(1/4 - 52\varepsilon)m$. Denote by Y_1 and Y_2 the vertex sets of these red component. Since $\Gamma_1 = \Gamma_B \cup \Gamma_R[X_1]$ has no perfect matching, it follows from Lemma 7.9 that X_2 contains an independent set of size at least $(1/2 - 50\varepsilon)m$, implying that $\Gamma[Y_1, Y_2]$ is almost empty and $\Gamma[X_1, X_2]$ is almost complete. It is easy to conclude from Lemma 7.9 that $\Gamma_B \cup \Gamma_R[Y_i]$ has a perfect matching for $i \in [2]$.

In order to complete the proof of Proposition 14.2 in this case, it remains to consider the case where Γ_1 has no perfect matching and X_2 contains a red component of order at least $(1/2 - 4\varepsilon)m$. The following claim completes the proof of Proposition 14.2.

Claim 14.6. *One of the following conditions holds.*

1. Either Γ_1 or Γ_2 has a perfect matching.
2. There exist disjoint sets $A_i, B_i \subseteq X_i$ of size at least $(1/4 - 500\varepsilon)m$ such that
 - $\Gamma_R[A_i \cup B_i]$ is connected and non bipartite for $i \in [2]$.
 - $\Gamma_B[A_i \cup B_{3-i}]$ is connected for $i \in [2]$

Proof. Without loss of generality, we assume that $|X_1| \geq |X_2|$. We assume that both Γ_1 and Γ_2 have no perfect matchings and make use of Lemma 7.9 (with parameter 50ε). We start by considering Γ_1 . It is easy to verify that Γ_1 is 2-connected. Furthermore, since the vertices of X_1 have degree at least $(3/4 - 2\varepsilon)$, it follows that the two last conditions cannot hold (because they imply that there are many vertices in X_1 of degree at most approximately $m/2$). We conclude that the following holds.

There exists an independent set $A_2 \subseteq X_2$ of size at least $(1/4 - 200\varepsilon)m$ such that $|N_{\Gamma_1}(A_2)| \leq (1/4 + 150\varepsilon)m$.

Similarly, if Γ_2 has no perfect matching, one of the two following conditions holds.

1. X_1 contains an independent set Y_1 of size at least $(1/2 - 50\varepsilon)m$.
2. There exists an independent set $A_1 \subseteq X_1$ of size at least $(1/4 - 200\varepsilon)m$ such that $|N_{\Gamma_2}(A_1)| \leq (1/4 + 150\varepsilon)m$.

Suppose that Condition (1) above holds. Then every vertex in A_2 has at least $(1/2 - 152\varepsilon)m$ red neighbours (in Γ) in X_2 . Denote

$$\begin{aligned} B_1 &= Y_1 \cap N_{\Gamma_1}(A_2) \\ A_1 &= Y_1 \setminus N_{\Gamma_1}(A_2) \\ Y_2 &= N_{\Gamma_R}(A_2) \\ B_2 &= Y_2 \cap N_{\Gamma_2}(A_1) \end{aligned}$$

It is easy to verify that Condition 2 from Claim 14.6 holds, completing the proof in this case.

We now suppose that Condition (2) holds. So we have sets $A_i \subseteq X_i$ of size at least $(1/4 - 200\varepsilon)m$ such that $|N_{\Gamma_{3-i}}(A_i)| \leq (1/4 + 150\varepsilon)m$. We write $N(A_i)$ for $N_{\Gamma_{3-i}}(A_i)$ as a shorthand.

If $A_1 \not\subseteq N(A_2)$, denote $A'_1 = A_1 \setminus N(A_2)$ and pick $u \in A'_1$. Then $N(u, X_2) \subseteq N_{\Gamma_2}(A'_1)$ and $N(u, X_2) \cap A_2 = \emptyset$. Since $\deg(u, X_2) \geq (1/4 - 6\varepsilon)$, we conclude $|N_{\Gamma_2}(A'_1) \setminus A_2| \geq (1/4 - 6\varepsilon)m$. In particular, $A_2 \not\subseteq N(A_1)$, so by the same argument, $|N_{\Gamma_1}(A'_2) \setminus A_1| \geq (1/4 - 5\varepsilon)m$, where $A'_2 = A_2 \setminus N(A_1)$. Define

$$B_i = N_{\Gamma_i}(A'_{3-i}) \cap N_{\Gamma_R}(A'_i).$$

It is easy to check that Condition (2) in Claim 14.6 holds.

Finally, if $A_1 \subseteq N(A_2)$ and $A_2 \subseteq N(A_1)$ denote $B_i = X_i \setminus (N(A_1) \cup N(A_2))$. There are no edges in Γ between A_1 and B_2 or between A_2 and B_1 . It is easy to conclude from here that Condition (2) in Claim 14.6 holds. \square

\square

The proof of Lemma 14.1 concludes our proof of Theorem 1.1. We remind the reader that Lemmas 7.13 and 7.14 were used several times in the proof, and we have yet to prove them. The next two sections, Sections 15 and 16 are devoted to the proofs of Lemmas 7.13 and 7.14 respectively.

15 Proof of Lemma 7.13

In this section, we prove Lemma 7.13. Before turning to the proof, we remind the reader of the statement.

Lemma (7.13). *Let $\frac{1}{n} \ll \varepsilon \ll 1$ and let G be a graph on n vertices with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that $S, T \subseteq V(G)$ satisfy the following conditions.*

- S, T are disjoint and $|S|, |T| \geq (1/2 - \varepsilon)n$.
- $\delta(G_B[S, T]) \geq (1/4 - \varepsilon)n$.
- For every $S' \subseteq S, T' \subseteq T$ with $|S'|, |T'| \geq (1/4 - 100\varepsilon)n$, we have $e(G_B[S', T']) \geq 25\varepsilon n^2$.

Then $V(G)$ may be partitioned into a red cycle and a blue one.

The main tool we use in the proof is Lemma 7.10, which is a stability version of a special case of Chvátal's theorem, Theorem 7.1. Our aim would be to find a short red cycle C and a short blue path P , whose removal from G leaves a balanced bipartite graph. We then apply Lemma 7.10 to deduce that P may be extended to a blue cycle with vertex set $V(G) \setminus V(C)$.

Proof of Lemma 7.13. We start by modifying the sets S, T as follows.

$$\begin{aligned} S_1 &= S \cup \{v \in V(G) \setminus (S \cup T) : \deg_B(v, T) \geq 24\varepsilon n\}. \\ T_1 &= T \cup \{v \in V(G) \setminus (S_1 \cup T) : \deg_B(v, S) \geq 24\varepsilon n\}. \\ X &= V(G) \setminus (S_1 \cup T_1). \end{aligned}$$

Remark. The vertices in X have red degree at least $(3/4 - 50\varepsilon)n$ in G .

We will find a red cycle and a blue path with one end in S and one in T , which are disjoint, cover X and their removal from G leaves a balanced bipartite graph with a large number of vertices. We will then use the following claim to obtain the required partition into a blue cycle and a red one.

Claim 15.1. *Let $S' \subseteq S_1, T' \subseteq T_1$ be such that $|S'| = |T'| \geq (1/2 - 12\varepsilon)n$. Then $G_B[S', T']$ is Hamiltonian. Furthermore, for every $s \in S'$ and $t \in T'$, the graph $G_B[S', T']$ contains a Hamiltonian path with ends s and t .*

Proof. Denote $G' = G_B[S', T']$ and $Y = (S' \cup T') \setminus (S \cup T)$. We claim that there exists a path P of length at most $12\varepsilon n$ whose vertex set contains Y . Indeed, we may construct P greedily, by adding a vertex of Y one at a time. Suppose that we want to add the vertex $y_1 \in Y$ to a path P in G' of length at most $12\varepsilon n$, one of whose ends is $y_2 \in Y$. We may pick $z_1, z_2 \in ((S \cup T) \cap V(G')) \setminus V(P)$ such that z_i is a neighbour of y_i in G' . We have $N_{G'}(z_i) \geq (1/4 - 11\varepsilon)n$. In particular, by the third assumption of the lemma, there exists a path of length at most 2 between $N_{G'}(z_1)$ and $N_{G'}(z_2)$. Thus we may add y_1 to P using at most five additional vertices. Using this process, we obtain the desired path P , containing the vertices of Y . Denote by s, t the ends of P and assume that $s \in S, t \in T$ (we may need to extend P slightly).

Let G'' be the graph obtained from G' by removing the inner vertices of P and denote $n'' = |G''|$ and $\eta = 25\varepsilon$. Then $\delta(G'') \geq (1/4 - \eta)n''$. Let $S' \subseteq S, T' \subseteq T$ be subsets of size at least $(1/4 - 3\eta)n'' \geq (1/4 - 75\varepsilon)(1 - 20\varepsilon)n \geq (1/4 - 80\varepsilon)n$. Then by the assumptions of Lemma 7.13, $e(G'[S', T']) \geq \eta n^2$. By Lemma 7.10, it follows that G' is Hamiltonian. The same argument may be used to show that G' contains a Hamilton path with ends s, t for every $s \in S', t \in T'$. \square

We consider several cases, depending on the size of X and the behaviour of the vertices in X .

Case 1: $X = \emptyset$

Without loss of generality, $|S_1| \geq |T_1|$. Denote $k = |S_1| - |T_1|$. Suppose first that k is even. We use the following claim.

Claim 15.2. *The graph $G[S_1]$ either contains a blue path of length k , or it contains a red cycle of length k .*

Proof. Suppose that $G_B[S_1]$ has no path of length k . It follows from Erdős and Gallai's theorem, Theorem 7.3, that $e(G_B[S_1]) \leq kn/2 \leq \varepsilon n^2$ (note that $k \leq 2\varepsilon n$). Setting $\eta = \sqrt{\varepsilon}$, there are at most ηn vertices in S_1 of blue degree at least ηn in S_1 . Denote by U the set of vertices in S_1 with blue degree at most ηn . Then $\delta(G_R[U]) \geq (1/4 - 2\eta)n$.

We show that $G_R[U]$ contains a cycle of length k . We may assume $k \geq 4$ because this assertion is trivial for $k = 0, 2$. Pick $u \in U$ and denote $A = N_R(u) \cap U$ (so $|A| \geq (1/4 - 2\eta)n$). If $G_R[A]$ has a path of length $k - 2$, together with the vertex u it forms a red cycle in S of length k .

Thus, we assume that $G_R[A]$ contains no path of length $k - 2$. It follows from Theorem 7.3 that $e(G_R[A]) \leq |A| \cdot \varepsilon n$. We deduce that at most $|A|/2$ vertices in A have red degree at least $4\varepsilon n$ in A . In particular, we may pick a set B of $k/2$ vertices in A with red degree at most $4\varepsilon n$ in A . For every $v \in B$ we have $\deg_R(v, U \setminus A) \geq (1/4 - 4\varepsilon - \eta)n$. It follows that every two vertices in B have at least say $n/8$ common red neighbours in $U \setminus A$. In particular, if $B = \{b_1, \dots, b_{k/2}\}$, we may pick distinct $c_1, \dots, c_{k/2} \in U \setminus A$ such that $(b_1, c_1, \dots, b_{k/2}, c_{k/2})$ is a red cycle in S of length k . \square

By Claim 15.2, either S_1 contains a blue path P of length k or it contains a red cycle C of length k . In the first case, it is easy to verify that P may be extended to a Hamilton cycle of G_B by Claim 15.1. Indeed, consider the bipartite graph $G_B[S_1 \setminus U, T_1]$ where U is the set of inner vertices of P_1 . This graph is almost balanced (namely the first set has one more vertex than the other), so it contains a Hamilton path whose ends are the ends of P . In the second case, we may conclude directly from Claim 15.1 that the graph $G_B \setminus V(C)$ is Hamiltonian.

We now suppose that k is odd. If S_1 contains a blue path P of length k , we continue as before. Otherwise, if S_1 contains a blue edge uv , we may find a red cycle C in $S_1 \setminus \{u, v\}$ of length $k - 1$, by the argument of Claim 15.1. It follows that the graph $G_B \setminus V(C)$ is Hamiltonian. Finally, if S_1 has no blue edges, we have $\delta(G_R[S_1]) \geq |S_1| - n/4 > |S_1|/2$, since $|S_1| \geq (n + 1)/2$. It follows from Bondy's theorem (7.4) that $G_R[S_1]$ is pancyclic, in particular it contains a cycle C of length k . We proceed as before to conclude that $G_B \setminus V(C)$ is Hamiltonian.

Case 2: $|X| = 1$

Denote $X = \{x\}$. Again, we assume that $|S_1| \geq |T_1|$ and denote $k = |S_1| - |T_1|$. Suppose first that k is odd. We shall use the following claim, whose proof is similar to the proof of Claim 15.2.

Claim 15.3. *Either $G[S_1]$ has a blue path P of length k , or $G[S_1 \cup \{x\}]$ has a red cycle C of length $k + 1$ containing x .*

Proof. If $G[S_1]$ contains no blue path of length k , we consider the set $A = N_R(x, S_1)$. Recall that $|A| \geq (1/4 - 25\varepsilon)n$. As in Claim 15.2, we conclude that either $G[A]$ contains a red path of length $k - 1$, or $G[A, S_1 \setminus A]$ contains a red path of length $k - 1$ with both ends in A . \square

If $G[S_1]$ contains a path of length k , the graph $G_B \setminus \{x\}$ is Hamiltonian and we may take (x) to be the red cycle. Otherwise, let C be a red cycle of length $k + 1$ in $S_1 \cup \{x\}$ containing x .

We now consider the case where k is even. Note that if $k = 0$, we can take (x) to be the red cycle and $G_B[S_1, T_1]$ is Hamiltonian. Thus, we assume that $k \geq 2$. Denote $A = N_R(x, S_1)$.

Claim 15.4. *One of the following conditions holds.*

1. $G[S_1]$ has a blue path of length k .
2. $G[S_1 \cup \{x\}]$ contains a red cycle C of length k going through x and a blue edge e which is disjoint of C .
3. $G[S_1 \cup \{x\}]$ has a red cycle C of length $k + 1$ going through x .
4. $G[S_1]$ has no blue edges and $G[A]$ has no red edges.

Proof. It is easy to conclude, as in Claims 15.2 and 15.3, that if $G[S_1]$ has at least one blue edge, one of the first two conditions holds. Thus we assume that $G[S_1]$ has no blue edges. Assuming the fourth condition does not hold, we may further assume that $G[A]$ has a red edge uv . We prove that the third condition holds under these assumptions. As before, we may assume that $e(G_R[A]) \leq \varepsilon n^2$, because otherwise $G[A]$ contains a red path of length $k - 1$ and the third condition holds.

Note that we have $\delta(G_R[S_1]) \geq |S_1| - n/4 \geq n/4$. Thus we may construct a red path in $G[S_1]$ on $k - 2$ vertices $P = (v = v_1, \dots, v_{k-2})$. If there exists a vertex $v_{k-1} \in S_1$ which is a common red neighbour of v_{k-2} and x , we obtain the red cycle $C = (v_1, \dots, v_{k-1}, x, u)$. Otherwise, the sets $A' = A \setminus (V(P) \cup \{u\})$ and $B = N_R(v_{k-2}, S) \setminus (V(P) \cup \{u\})$ are disjoint. Note that $|A'|, |B| \geq (1/4 - 60\varepsilon)n$. Since $e(G_R[A']) \leq \varepsilon n^2$ and $\delta(G_R[S_1]) \geq n/4$, we conclude that $G_R[A', B]$ contains an edge v_{k-1}, v_k (where $v_{k-1} \in B, v_k \in A'$). It follows that (v_1, \dots, v_k, x) is a red cycle in S_1 of length k . \square

In each of the first three conditions of Claim 15.4, we may proceed as before to conclude that G has the desired partition into a red cycle and a blue one. Thus we may assume that S_1 spans no blue edges and A spans no red edges. In particular, $\deg_R(x, S_1) \leq n/4$, hence x has at least two blue neighbours in S_1 , since $\deg(x, S_1) \geq |S_1| - (n/4 - 1) \geq n/4 + 1/2 + k/2 \geq n/4 + 3/2$. We consider the cases $k = 2$, $k = 4$ and $k \geq 6$ separately.

Suppose first that $k = 2$. Let $u, v \in S_1$ be two blue neighbours of x in S_1 . Let C_1 be a (red) cycle consisting of a single vertex in $S_1 \setminus \{u, v\}$. We may find a blue cycle spanning $V(G) \setminus V(C_1)$ by Claim 15.1.

Suppose now that $k = 4$. We have $|S_1| = n/2 + 3/2$. Thus x has at least three blue neighbours $u, v, w \in S_1$. Pick an edge ab in S_1 such that a and b are distinct from u, v, w . Since $\deg_R(a, S_1) \geq |S_1| - n/4 \geq n/4 + 3/2$, we conclude that a and b have a common red neighbour $c \in S_1$. Let C_1 be the red triangles (abc) . Without loss of generality $c \neq u, v$. We proceed as before, to show that the graph $G_B \setminus \{a, b, c\}$ is Hamiltonian.

It remains to consider the case $k \geq 6$. Fix u, v to be blue neighbours of x in S_1 and denote $S_2 = S_1 \setminus \{u, v\}$. Note $|S_2| \geq n/2 + 1/2$ and $\delta(G_R[S_2]) \geq |S_2| - n/4 > |S_2|/2$. It follows from Theorem 7.4, that $G_R[S_2]$ is pancyclic. In particular, it contains a red cycle of length $k - 1$. We proceed as before.

From now on, we may assume that $|X| \geq 2$. Set $\eta = 2\sqrt{\varepsilon}$.

Case 3: there exist $x_1, x_2 \in X$ with $\deg_R(x_1, S), \deg_R(x_2, T) \geq (1/4 + 10\eta)n$

Denote $X = \{x_1, \dots, x_r\}$. Recall that $\deg_R(x_i) \geq (3/4 - 50\varepsilon)n$. Thus we may pick distinct $y_i \in V(G) \setminus X$ such that y_i is a common red neighbour of x_i and x_{i+1} for $i \in [2, r - 1]$. Denote $Y = \{y_2, \dots, y_{r-1}\}$. Let $S_2 = S_1 \setminus Y$ and $T_2 = T_1 \setminus Y$.

Claim 15.5. *The graph $G[S_2]$ either contains a blue path of length $5\varepsilon n$ or for every $2 \leq l \leq 5\varepsilon n$ it contains a red path of length $l - 1$ whose one end is a red neighbour of x_1 and the other is a red neighbour of x_r .*

Similarly, the graph $G[T_1]$ either contains a blue path of length $5\varepsilon n$ or for every $2 \leq l \leq 5\varepsilon n$ it contains a path of length $l - 1$ a red neighbour of x_1 as one end and a red neighbour of x_2 as the other end.

Proof. We prove the first part of Claim 15.5, the second part of follows similarly. Suppose that $G[S_2]$ has no blue path of length $5\varepsilon n$. It follows by Theorem 7.3 that $e(G[S_2]) \leq 3\varepsilon n^2$. Denote by U the set of vertices in S' with blue degree at most ηn (where $\eta = 2\sqrt{\varepsilon}$). Then $|S_2 \setminus U| \leq \eta n$ and $\delta(G_R[U]) \geq (1/4 - 3\eta)n$. Pick $u \in N_R(x_r, U)$. Greedily construct a red path $P = (u = u_1, \dots, u_{l-1})$ in U . Denote $A = N_R(x_1, U \setminus V(P))$ and $B = N_R(u_{l-1}, U \setminus V(P))$. As $|Y| + |P| \leq \eta n$, we have $|A| \geq (1/4 + 6\eta)n$ (by the assumption on x_1) and $|B| \geq (1/4 - 4\eta)n$. Thus $|A \cap B| \geq |A| + |B| - |U| \geq \eta n$. It follows that we may pick $u_l \in A \cap B$. The path $(x_r, u_1, \dots, u_l, x_1)$ satisfies the requirements of Claim 15.5. □

Without loss of generality, we assume that $|S_2| \geq |T_2|$. Denote $k = |S_2| - |T_2|$, so $k \leq 4\varepsilon n$. It is easy to conclude from Claim 15.5 that we may find vertex-disjoint paths $P_1 \in G_B[S_2]$ and $Q_1 \in G_R[S_2]$ such that the following holds.

- Either P_1 has length k and Q_1 is a singleton, or P_1 is the empty path and Q_1 has length k .
- One end of Q_1 is a red neighbour of x_1 and the other end is a red neighbour of x_r (if Q_1 is a singleton, then it is a common red neighbour of x_1, x_r).

Indeed, if $G[S_2]$ contains a blue path of length $5\epsilon n$, we may find such P_1, Q_1 where P_1 has length $k - 1$ and Q_1 is any common red neighbour of x_1, x_r in S_2 (note that such a common neighbour exists). Otherwise, $G[S_2]$ contains a red path of length $k - 1$ whose ends are a red neighbour of x_1 and a red neighbour of x_2 .

Similarly, we may pick P_2, Q_2 to be a blue and a red path in $G[T_2]$ as follows.

- Either P_2 is the empty path and Q_2 has length 1 or P_2 has length 1 and Q_2 is a singleton.
- Q_2 has one end which is a red neighbour of x_1 and the other is a red neighbour of x_2 .

We take C to be the red cycle $(Q_1 x_1 Q_2 x_2 y_2 x_3 \dots x_l)$. The paths P_1, P_2 may be extended to a Hamilton cycle of $G_B \setminus V(C)$ by Claim 15.1.

Without loss of generality, we may now assume the following.

$$\deg_R(x, T) \leq (1/4 + 10\eta)n \text{ for every } x \in X. \quad (2)$$

It follows that $\deg(x, S) \geq (1/2 - 11\eta)n$ for every $x \in X$.

Case 4: some $x_1, x_2 \in X$ have at least $3\eta n$ common red neighbours in T

We proceed similarly to the previous case. For $2 \leq i \leq r - 1$, pick y_i to be a common red neighbour of x_i and x_{i+1} such that the y_i 's are distinct and do not belong to X .

Let $S_2 = S_1 \setminus Y$ and $T_2 = T_1 \setminus Y$. Note that $\deg_R(x_i, S_1) \geq (1/2 - 11\eta)n$ for $i \in [r]$ by (2). Consider the set D of common red neighbours of x_1, x_r in S' . Then $|D| \geq (1/2 - 23\eta)n$. Clearly, D has either a blue path or a red one of length $5\epsilon n$.

Claim 15.6. *The graph $G[T_2]$ either has a blue path of length $5\epsilon n$ or it contains a red path of length l with ends which are neighbours of x_1, x_2 respectively, for every even $2 \leq l \leq 5\epsilon n$.*

Proof. Suppose that $G[T_2]$ has no blue path of length $5\epsilon n$. Denote by U the set of vertices of T_2 with at most ηn blue neighbours. Then $|T_2 \setminus U| \leq \eta n$ and $\delta(G_R[U]) \geq (1/4 - 3\eta)n$. Denote $A = N_R(x_1, U)$ and $B = N_R(x_2, U)$. Note that $|A|, |B| \geq (1/4 - 2\eta)n$ and $U \cap A \cap B \neq \emptyset$, by the assumptions on x_1 and x_2 , thus we may pick $u_1 \in U \cup A \cup B$.

If $|A \cup B| \geq (1/4 + 10\eta)n$, we may find a red path of length l as follows. Greedily pick a red path $P = (u_1, \dots, u_l)$ in U . As in Claim 15.5, there exists $u_{l+1} \in (A \cup B) \setminus V(P)$ which is a red neighbour of u_l . The path (u_1, \dots, u_{l+1}) satisfies the requirements.

If $|A \cup B| \leq (1/4 + 10\eta)n$, it follows that $|A \cap B| \geq |A| + |B| - |A \cup B| \geq (1/4 - 14\eta)n$. If $A \cap B$ contains a red path of length l , we are done. Otherwise, we may continue as in Claim 15.2 to conclude that the graph $G_R[A \cap B, U \setminus (A \cap B)]$ has a path of length l with ends in $A \cap B$. \square

Denote $k = |S_2| - |T_2|$. Pick $2 \leq l_1, l_2 \leq 5\epsilon n$ such that l_2 is even and $l_1 - l_2 = k$. Similarly to the previous case, we may pick vertex-disjoint paths $P_1 \in G_B[S_2]$, $Q_1 \in G_R[S_2]$, $P_2 \in G_B[T_2]$ and $Q_2 \in G_R[T_2]$ with the following properties.

- Q_1 has ends which are a red neighbour of x_1 and a red neighbour of x_r . Similarly, Q_2 has ends which are a red neighbour of x_2 and a red neighbour of x_1 .
- Either $P_i = \emptyset$ and Q_i has length l_i or P_i has length l_i and Q_i is a singleton, for $i \in [2]$.

As before, we take C to be the red cycle $(Q_1x_1Q_2x_2y_2x_3 \dots x_r)$. The paths P_1, P_2 may be extended to a Hamilton cycle in $G_B \setminus V(C)$.

Note that if $|X| \geq 3$, one of Cases 3 and 4 holds (perhaps with the roles of S and T reversed). Thus we may assume that $|X| = 2$, and denote $X = \{x_1, x_2\}$.

Case 5: $\deg_R(x_i, T) \leq (1/4 + 10\eta)n$ and $|N_R(x_1, T) \cap N_R(x_2, T)| \leq 3\eta n$

Denote

$$A = N_R(x_1, T_1), B = N_R(x_2, T_1) \text{ and } D = N_R(x_1, S_1) \cap N_R(x_2, S_1).$$

Note that $|D| \geq (1/2 - 23\eta)n$.

If $|S_1| \geq |T_1| + 2$, denote $k = |S_1| - |T_1|$, so $2 \leq k \leq 2\epsilon n$. As usual, $G[D]$ either contains a blue path P or a red path Q of length $k - 2$. In the former case, pick C to be a 4-cycle consisting of x, y and two vertices in $D \setminus V(P)$. In the latter case, extend Q to a cycle C through x, y using an additional vertex of D . For convenience denote $P = \emptyset$. In both cases, the path P may be extended to a Hamilton cycle of $G_B \setminus V(C)$. From now on, we may assume the following.

$$|S_1| \leq |T_1| + 1.$$

Denote $k = |T_1| - |S_1|$ (so $-1 \leq k \leq 2\epsilon n$). A *path forest* is a collection of vertex disjoint paths. If $G_B[T_1]$ contains a path forest H with $k + 2$ edges, we can finish the proof as follows. Pick C to be any red 4-cycle consisting of x_1, x_2 and two vertices from D . Then $G_B \setminus V(C)$ has a Hamilton cycle extending H . This can be seen by connecting the paths of H with paths in $G_B[S_1, T_1]$ of length at most 6 and using Claim 15.1. Thus we may assume the following.

$$G_B[T_1] \text{ has no path forest with } k + 2 \text{ edges. In particular, } e(G_B[T_1]) \leq \epsilon n^2. \quad (3)$$

Suppose that $G_R[T_1]$ contains a path P with one end in A and the other in B , of length l , where $0, k \leq l \leq 3\epsilon n$ (by a path of length 0 we mean a single vertex). Then we may find the desired cycle partition as follows. $G[D]$ contains a path Q of length $l - k$ which is either red or blue. If Q is red, take C to be the red cycle (x_1Px_2Q) , and the remaining graph $G_B \setminus V(C)$ is Hamiltonian. If Q is blue, take $C = (x_1Px_2u)$ where $u \in D \setminus V(Q)$. The leftover graph $G_B \setminus V(C)$ has a Hamilton path extending Q . Thus from now on we assume the following.

$$G_R[T_1] \text{ has no path of length } l, \text{ where } 0, k \leq l \leq 3\epsilon n, \text{ with ends in } A \text{ and } B. \quad (4)$$

In particular, $e(G_R[A, B \setminus A]) \leq \epsilon n^2$, implying that $G_R[A], G_R[B]$ are almost complete (recall that $A \cap B$ have a small intersection). Suppose that x_1 has two blue neighbours $u, v \in T_1$. It follows from the previous assumption that $G_R[B]$ contains a path P on $k + 1$ vertices. Form a red cycle C by adding the vertex x_2 to P . Then the graph $G_B \setminus V(C)$ is Hamiltonian, since the

graph $G_B \setminus (V(C) \cup \{x_1\})$ has a Hamilton path with ends u, v . Thus we may assume that both x_1 and x_2 have at most one blue neighbour in T_1 . In particular,

$$|A|, |B| \geq |T_1| - n/4. \quad (5)$$

We can now finish the proof in case $|S_1| - |T_1| \in \{0, 1\}$. Note that we have $e(G_B[T_1]) \leq 2$ by Assumption (3). It is easy to conclude, using Assumption (14) that one of the following conditions holds.

- $A \cap B \neq \emptyset$.
- $A \cap B = \emptyset$ and $e(G_R[A, B]) \geq 1$.
- There exists a vertex $u \in T_1$ which has red neighbours in both A and B .

It follows that there exists a red path in T_1 with one end in A and the other in B of length at most 2, contradiction Assumption (4). Thus the desired monochromatic cycle partition exists.

We may now assume $k = |T_1| - |S_1| \geq 1$. Denote $Y = \{u \in T_1 : \deg_B(u, T_1) \geq \eta n\}$. Then

$$|Y| \leq (k + 1)/2 \quad (6)$$

because otherwise $G_B[T_1]$ contains at least $(k+2)/2$ vertex-disjoint paths of length 2, contradicting Assumption (3).

Denote $A' = A \setminus Y$ and $B' = B \setminus Y$. If there exists a path of length at most 2 in $G_R[T_1]$ with one end in A' and the other in B' , this path may be extended to a red path between A' and B' of length $k + 2$, contradicting assumption (4). Thus we assume the following.

$$A' \cap B' = \emptyset. \quad (7)$$

$$e(G_R[A', B']) = 0. \quad (8)$$

$$\text{No vertex in } T_1 \text{ has red neighbours in both } A' \text{ and } B'. \quad (9)$$

It is not hard to reach a contradiction from here, thus finishing the proof. By (7), $A \cap B \subseteq Y$. In particular,

$$|Y| \geq |A \cap B| \geq |A| + |B| - |T_1| \geq |T_1| - n/2 = (k - 2)/2. \quad (10)$$

Also, by (6),

$$|A'|, |B'| \geq |T_1| - n/4 - |Y| \geq n/4 - 3/2. \quad (11)$$

If $|Y| \leq (k - 1)/2$, it follows that $|A'|, |B'| \geq n/4 - 1/2$. By the minimum degree condition, we have that $\delta(G[A', B']) \geq 1$, and it is easy to deduce that $G[A', B']$ has a (blue) path forest on at least four edges. By (10), we may complete this into a blue path forest in T_1 with at least $k + 2$ edges, a contradiction to (3).

Thus we assume that $|Y| \geq k/2$. It follows from (3) that

$$e(G_B \setminus Y) \leq 1. \quad (12)$$

If $|A'| > n/4 - 1$, the graph $G[A', B']$ has at least two (blue) edges, a contradiction. Thus we assume that $|A'|, |B'| \leq n/4 - 1$. It follows that $|A \cup B| \leq |A'| + |B'| + |Y| < |T_1|$. Pick

$u \in T_1 \setminus (A \cup B)$. By (11, 8, (12)), all but at most two vertices of $A' \cup B'$ are connected to u , contradicting (9). □

16 Proof of Lemma 7.14

In this section, we prove Lemma 7.14. We first remind the reader of the statement.

Lemma (7.14). *Let $\frac{1}{n} \ll \varepsilon \ll 1$ and let G be a graph on n vertices with $\delta(G) \geq 3n/4$ and a 2-colouring $E(G) = E(G_B) \cup E(G_R)$. Suppose that there exists a partition $\{S, T, X\}$ of $V(G)$ with the following properties.*

- $|S|, |T| \geq (1/2 - \varepsilon)n$.
- $|X| \leq 2$ and if $|X| = 2$, there exists $u \in X$ such that $\deg_R(x, S) \leq \varepsilon n$ or $\deg_R(x, T) \leq \varepsilon n$.
- The sets S and T belong to different components of $G_R \setminus X$.

Then $V(G)$ may be partitioned into a red cycle and a blue one.

The idea of the proof is as follows. By Lemma 7.13, we may assume that there exist subsets $S' \subseteq S, T' \subseteq T$ of size almost $n/4$ such that $G[S', T']$ is close to being empty, implying that the subgraphs $G_B[S', T \setminus T']$ and $G_B[S \setminus S', T']$ are almost complete. We aim, similarly to the proof of Lemma 7.13 to find a red cycle C , and two blue paths P_1, P_2 , whose removal from G leaves two balanced bipartite subgraphs of the aforementioned graphs. We then find Hamilton paths in the remainder subgraphs which together with P_1, P_2 form a blue cycle with vertex set $V(G) \setminus V(C)$. We remark that we run into some technical difficulties when X is non-empty.

Proof of Lemma 7.14. We abuse notation and slightly change the definition of S, T and X as follows. Denote

$$X = \{x, y\} \tag{13}$$

(if $|X| < 2$, add vertices to X arbitrarily). Recall that by the conditions of Lemma 7.14, we may assume that $\deg_R(x, S) \leq \varepsilon n$ or $\deg_R(x, T) \leq \varepsilon n$. If the former holds, we move x from X to T , otherwise we move x from X to S . Similarly, if $\deg_B(y, S) \geq n/32$ we move y from X to T and otherwise, if $\deg_R(y, T) \leq n/32$, we put y in S . After the modification, we have either $X = \emptyset$, or $X = \{y\}$ and $\deg_B(y) \leq n/16$. Furthermore, the only red edges in $G[S, T]$ are adjacent to x or y .

We may assume that there exist subsets $S' \subseteq S, T' \subseteq T$ of size at least $(1/4 - 100\varepsilon)n$ such that $e(G[S', T']) \leq 25\varepsilon n^2$, because otherwise the proof can be completed immediately by Lemma 7.13. It is easy to deduce the following claim (we omit the exact details of the proof).

Claim 16.1. *The following holds for some $\eta = \eta(\varepsilon) \geq \varepsilon$. There exist partitions $\{S_1, S_2\}$ of S and $\{T_1, T_2\}$ of T with the following properties.*

- $|S_1|, |S_2|, |T_1|, |T_2| \geq (1/4 - \eta)n$.
- All but at most ηn vertices of $S_2 \cup T_1$ have degree at most ηn in $G_B[S_2, T_1]$.

- The graphs $G_B[S_i, T_i]$ have minimum degree at least $n/64$. Furthermore, all but at most ηn vertices in these graphs have degree at least $(1/4 - \eta)n$.
- All but at most ηn vertices in the graphs $G[S_1, S_2]$ and $G[T_1, T_2]$ have degree at least $(1/4 - \eta)n$.

We shall also use the following claim which may be easily verified by Theorem 7.1.

Claim 16.2. *Let $S' \subseteq S_i$ and $T' \subseteq T_i$ be sets of equal size such that $|(S_i \cup T_i) \setminus (S' \cup T')| \leq 10\eta n$. Then $H = G_B[S', T']$ is Hamiltonian. Furthermore, for every $s \in S', t \in T'$, H contains a Hamilton path with ends s and t .*

Without loss of generality, we assume $|S| \geq |T|$. Denote $k = |S| - |T|$. We consider two cases according to the size of X .

Case 1: $X = \emptyset$

Let A be the set of red neighbours of x in $G[S, T]$ and B the set of red neighbours of y in $G[S, T]$. Then

$$|A| \leq n/16 \text{ and } |B| \leq (1/2 - 1/64)n. \quad (14)$$

If $k \leq 1$, we may find a partition of $V(G)$ into a blue cycle and a red one as follows. Pick any red cycle C_1 in S of length k , so C_1 is either the empty set or a vertex. Denote $S' = S \setminus V(C_1)$ and consider the balanced bipartite graph $H = G_B[S', T]$. Then

$$\deg_{H_B}(u) \geq \begin{cases} n/64 & u \in \{x, y\} \\ (n - k)/2 - (n/4 - 1) - 2 = n/4 - k/2 - 1 & u \in A \\ n/4 - k/2 & u \in B \setminus A \\ n/4 - k/2 + 1 & \text{otherwise} \end{cases}$$

It follows by Theorem 7.1 that H is Hamiltonian (note that B is a subset of either S or T), implying that the desired partition of $V(G)$ into a red cycle and a blue exists.

Case 1.1: $|T_1| < |S_1|$ and $|T_2| < |S_2|$

Recall that $k \geq 2$, so

$$|S| = (n + k)/2 \geq n/2 + 1. \quad (15)$$

Claim 16.3. *The graph $G_B[S_1, T_2] \cup G_B[S_2, T_1]$ contain a (blue) matching of size 2.*

Proof. If $|S_2| > n/4 + 1$, every vertex in T_1 has at least three neighbours in S_2 , thus every vertex in $T_1 \setminus A$ has at least two blue neighbours in S_2 , implying that we may find the desired matching. Similarly, if $|S_1| > n/4 + 1$, the graph $G_B[S_1, T_2]$ contains a matching of size 2. Thus we may assume that $|S_1|, |S_2| \leq n/4 + 1$, implying that $|S_1|, |S_2| \geq n/4$.

Without loss of generality, $|S_1| \geq n/4, |S_2| > n/4$. Hence every vertex in $T_1 \setminus A$ has at least two neighbours in S_2 . If $y \notin S_2$, these two neighbours are both blue, and we may find the required matching in $G_B[S_2, T_1]$. If $y \in S_2$, we conclude similarly that both graphs $G[S_1, T_2]$ and $G[S_2, T_1]$ contain at least one blue edge, implying that the desired matching exists. \square

Without loss of generality, we shall assume the following.

$$s_1 t_1, s_2 t_2 \text{ is a blue matching in } G[S_2, T_1], \text{ where } s_1, s_2 \in S_2 \text{ and } t_1, t_2 \in T_1. \quad (16)$$

It is easy to verify that one of the following holds.

1. $G[S_1]$ contains a blue path of length $5\eta n$.
2. $G[S_1, T_2]$ contains a blue path of length $5\eta n$.
3. $G[S_1, S_2]$ contains a blue path of length $5\eta n$.
4. $G[S_1]$ contains a red cycle of length l for every $l \leq 5\eta n$, and $G[S_2]$ contains a blue path of length $5\eta n$.
5. For every $l_1, l_2 \leq 5\eta n$, $G[S_1, S_2]$ contains a red cycle with l_1 vertices from S_1 and l_2 vertices from S_1 .

Indeed, if the first three conditions do not hold, by Theorem 7.3, the graphs $G_R[S_1]$ and $G_R[S_1, S_2]$ are almost complete, so either the third condition holds or $G_R[S]$ is almost complete.

In Case (1), we conclude that $G[S_1 \cup T_1]$ contains a blue Hamilton path Q_1 with ends t_1, t_2 . Indeed, let P_1 be a path in $G_B[S_1]$ on $|S_1| - |T_1| + 2$ vertices (note that this quantity is smaller than $5\eta n$). By Claim 16.2, we may extend P_1 to a Hamilton path of $G_B[S_1 \cup T_1]$ with ends t_1 and t_2 . Now consider $G[S_2]$. Note that $G[S_2]$ is almost complete (as $G[S_2, T_1]$ is almost empty), thus it either contains a blue path of length $|S_2| - |T_1| - 2$ or a red cycle of length $|S_2| - |T_1| - 1$. In any case, we may partition $S_2 \cup T_2$ into a red cycle C and a blue path Q_2 with ends s_1, s_2 (where in the former case the red cycle is empty). By joining Q_1 and Q_2 using the edges $s_1 t_1$ and $s_2 t_2$, we obtain a blue cycle on the vertices $V(G) \setminus V(C)$.

In Case (2), let P_1 be a path in $G_B[S_1, T_2]$ with ends $s \in S_1, t \in T_2$ with exactly $|S_1| - |T_1| + 1$ vertices from S_1 . Denote by U the set of inner vertices of P_1 and let $S'_1 = S_1 \setminus U$ and $T'_2 = T_2 \setminus U$. By Claim 16.2 the graph $G_B[S'_1, T_1]$ has a Hamilton path Q_1 with ends s, t_1 . Consider the graph $G[S_2 \cup T'_2]$. We conclude as before that $G[S_1, T'_1]$ can be partitioned into a blue path with ends s_1, t and a red cycle, completing the required partition of $V(G)$ into a blue cycle and a red one.

We assume that Condition (2) does not hold, implying that $G[S_1, T_2]$ is almost empty and so $G[T_2]$ is almost complete. Suppose that Condition (3) holds. Let P_1 be a path in $G_B[S_1, S_2]$ with ends $s_3 \in S_1, s_4 \in S_2$ and exactly $|S_1| - |T_1| + 1$ vertices from S_1 . Define U to be the set of inner vertices of P_1 and let $S'_i = S_i \setminus U$. It can be shown as before that $G_B[S'_1, T_1]$ has a Hamilton path with ends s_3, t_1 and that $G_B[S'_2, T_2]$ can be partitioned into a blue path with ends s_1, s_4 and a red cycle. Note that it may happen that $|T_2| \geq |S'_2|$ in which case the red cycle is contained in T_2 (here we use the assumption that $G[T_2]$ is almost complete).

Now suppose that Condition (4) holds. It follows that $G[S_1 \cup T_1]$ may be partitioned into a blue path with ends t_1, t_2 and a red cycle, and that $G[S_2 \cup T_2]$ contains a blue Hamilton path with ends s_1, s_2 .

Finally, if Condition (5) holds, let C be a red cycle consisting of $|S_1| - |T_1| + 1$ vertices from S_1 and $|S_2| - |T_2| - 1$ vertices from S_2 . As before, the graphs $G[S_1 \setminus V(C), T_1]$ and $G[S_2 \setminus V(C), T_2]$ contain blue Hamilton paths with ends t_1, t_2 and s_1, s_2 respectively.

Case 1.2: $|S_1| = |T_1|$ or $|S_2| = |T_2|$

Suppose that $|S_1| = |T_1|$. Similarly to Claim 16.3, we claim that $G[S_2, T_1]$ contains a blue matching of size 2. Indeed, if either $|S_2| > n/4 + 1$ or $|T_1| = |S_1| > n/4 + 1$ we may find the required matching as in the proof of the claim. Otherwise, we have $n/4 \leq |S_2|, |S_1| \leq n/4 + 1$ and without loss of generality $|S_2| > n/4$ and $|T_1| = |S_1| \geq n/4$, and again we may find the required matching as in Claim 16.3.

Similarly, if $|S_2| = |T_2|$, it follows that $G[S_1, T_2]$ contains a blue matching of size 2 and we proceed as before.

Case 1.3: $|S_1| < |T_1|$

In order to obtain the required balanced subgraph of $G[S_1, T_1]$, we use the following claim.

Claim 16.4. *There exist vertex-disjoint paths Q_1, Q_2 in $G_B[S, T]$ satisfying the following properties.*

- $|Q_1|, |Q_2| \leq 20\eta n$.
- Q_i has one end in S_2 and the other in T_1 .
- Denote $U = V(Q_1) \cup V(Q_2)$. Then

$$|U \cap T_1| - |U \cap S_1| = |T_1| - |S_1| + 1.$$

Proof. Consider the bipartite graph $H = G_B[T_1, S']$ where $S' = S \setminus \{u\}$ for some fixed $u \in S_1$. Recall that $|S'| = |S| - 1 \geq n/2$ (15). We show that there exist two edge-disjoint matchings M_1, M_2 of size $|T_1| - |S_1| + 1$ in $G[S_2, T_1]$ whose union contains no cycles. To that end, we show first that H contains a matching saturating T_1 , by showing that H satisfies Hall's condition, namely that for every $W \subseteq T_1$, we have $|N_H(W)| \geq |W|$.

Recall that x, y are the vertices that were in X originally (13), and A, B are their red neighbourhoods in $G[S, T]$. We consider four ranges for the size of W .

- $|W| \leq 2$. Here Hall's condition holds trivially because the minimum degree of a vertex from T_1 is larger than 2.
- $3 \leq |W| \leq n/4 - 1$. Recall that every vertex in T_1 , except for possibly x and y , has degree at least $n/2 - (n/4 + 1) = n/4 - 1$ in H . Thus, in this case we have

$$|N_H(W)| \geq n/4 - 1 \geq |W|.$$

- $n/4 - 1 < |W| \leq n/4$. By the lower bound on the size of A , there exists $w \in W \setminus (A \cup \{x, y\})$. w has at most one red neighbour in S' (namely y), thus

$$|N_H(W)| \geq \deg_B(w, S') \geq n/2 - n/4 = n/4 \geq |W|.$$

- $|W| > n/4$. In this case, every vertex in $S' \setminus (A \cup \{x, y\})$ has a neighbour in W , so

$$|N_H(W)| \geq |S'| - (n/16 + 2) \geq |T_1| \geq |W|.$$

It follows that there exists a matching in $G[S'_1 \cup S_2, T_1]$ saturating T_1 . We take M_1 to be a sub-matching in $G_B[T_1, S_2]$ of size $|T_1| - |S_1| + 1 \leq 5\eta n$.

Consider the graph H' obtained from H by removing the edges of H spanned by $V(M_1)$. It is easy to check by a similar analysis that H' contains a matching saturating T_1 . Indeed, if $1 \leq |W| \leq 5\eta n + 2$, we clearly have $|N_H(W)| \geq |W|$. If $5\eta n + 2 < |W| \leq n/4 - 1$, we have $|N_H(W)| \geq n/4 - 1 \geq |W|$. If $n/4 - 1 < |W| \leq n/4$, we may pick a vertex $w \in W \setminus (A \cup V(M_1) \cup \{x, y\})$ and continue as above. Finally, if $|W| > n/4$, every vertex in $S' \setminus (V(M_1) \cup A \cup \{x, y\})$ has a blue neighbour in W , so $|N_H(W)| > |W|$. So there exists a matching M_2 in $G_B[T_1, S_2]$, edge-disjoint of M_1 , of size $|T_1| - |S_1| + 1$.

Let F be the graph $(V(G), E(M_1) \cup E(M_2))$. Note that by the choice of H' , F contains no cycles (in fact it contains no paths of length larger than 3), hence it is a collection of vertex-disjoint paths.

We obtain paths Q_1, Q_2 as follows. Start with the collection of paths in F . In each stage, if there are two paths with an end in S_2 , connect them using a path in $G_B[S_2, T_2]$ of length at most 4, such that it remains vertex-disjoint of all other paths constructed so far. Similarly, if there are two paths with an end in T_1 we join them using a path of length at most 4 in $G_B[S_1, T_1]$. Note that whenever there are at least three paths, we may continue the process. We stop when exactly two paths remain.

Denote $U = V(Q_1) \cup V(Q_2)$, and let r_i be the number of vertices in T_1 of degree i in F , where $i \in [2]$. Then

$$\begin{aligned} |U \cap T_1| - |U \cap S_1| &= r_2 + (r_1 - 2)/2 + 2 = (2r_2 + r_1)/2 + 1 \\ &= |E(F)|/2 + 1 = |T_1| - |S_1| + 2. \end{aligned}$$

By the definition of Q_1, Q_2 , the set of edges of $G_B[S_2, T_1]$ which are contained in one of Q_1, Q_2 is exactly $E(F)$, so in particular, it has even size. It follows that either both paths Q_1, Q_2 have one end in S_2 and the other in T_1 or one of them has both ends in S_2 and the other has both ends in T_1 . Recall that by the way the paths Q_1, Q_2 were constructed, the first and last edges in each of them are in $G_B[S_2, T_1]$. Hence, by either removing the first and last edges from one of the paths or by removing the first edge from each of them, we obtain paths Q'_1, Q'_2 satisfying the requirements of Claim 16.4. \square

Let Q_1, Q_2 be paths as in Claim 16.4. Denote by $s_i \in S_2, t_i \in T_1$ the ends of Q_i , let U' be the set of inner vertices of these paths, and let $S'_i = S_i \setminus U'$ and $T'_i = T_i \setminus U'$. It is easy to verify that $G_B[S'_1, T'_1]$ has a Hamilton path with ends t_1, t_2 and that $G[S'_2 \cup T'_2]$ may be partitioned into a blue path with ends s_1, s_2 and a red cycle (note that $|S'_2| \geq |T'_2|$, and we use the fact that $G[S_2]$ is almost complete).

Case 1.4: $|S_2| < |T_2|$

Here we have $|S_1| \geq |T_1| + 2$. If $G[S_1, T_2]$ contains a blue path of length $10\eta n$, we may proceed as in Condition (2) from Case 1.1. Otherwise, $G[S_1, T_2]$ has few edges, so $G[S_1]$ contains many

edges and we may proceed as in the previous case.

Case 2: $X = \{y\}$

We start by showing that if $k \leq 2$, we may partition $V(G)$ into a red cycle and a blue one. Consider $A = N_R(y, S)$ and recall that $|A| \geq n/16$. If $k = 0$, we pick the red cycle C_1 to be (y) . If $k = 1$, we pick the red cycle C_1 to be a red edge (yz) , where $z \in A$. If $k = 2$ and A contains a red edge uv , we pick the red cycle C_1 to be the triangle (uvy) .

In each of these cases, the remainder graph $H = G[S', T]$, where $S' = S \setminus V(C_1)$ is a balanced bipartite graph. To obtain the desired partition of $V(G)$ into a red cycle and a blue one, we show that H_B is Hamiltonian. Indeed, denote $B = N_{H_R}(x)$. Then

$$\deg_{H_B}(u) \geq \begin{cases} (n-k-1)/2 - (n/4-1) - n/16 = 3n/16 + 1/2 - k/2 & v = x \\ (n-k-1)/2 - (n/4-1) - 1 = n/4 - k/2 - 1/2 & v \in B \\ (n-k-1)/2 - (n/4-1) = n/4 - k/2 + 1/2 & \text{otherwise} \end{cases}$$

It follows from Theorem 7.1 that indeed, H_B is Hamiltonian.

It remains to consider the case where $k = 2$ and A contains no red edges. Suppose first that y has two blue neighbours in S . Consider the graph $H = G[S', T']$, where $S' = S \setminus \{u\}$ for some $u \in S$ and $T' = T \cup \{y\}$. We claim that H_B is Hamiltonian. The desired partition may thus be obtained by letting (u) be the red cycle. Indeed, let $B = N_{H_R}(x)$. Then

$$\deg_{H_B}(v) \geq \begin{cases} 2 & v = y \\ (n-1)/2 - n/4 - n/16 = 3n/16 - 1/2 & v = x \\ (n-1)/2 - (n/4+1) = n/4 - 3/2 & v \in B \\ (n-1)/2 - n/4 = n/4 - 1/2 & v \in S' \setminus (B \cup \{x, y\}) \\ (n-1)/2 - (n/4-1) = n/4 + 1/2 & v \in T' \setminus (B \cup \{x, y\}) \end{cases}$$

By Theorem 7.1, H_B is indeed Hamiltonian.

We may now assume that y has at most one blue neighbour in S . It follows that $\deg_R(y, S) = |A| \geq |S| - n/4 = n/4 + 1/2$. Hence, $G[A]$ contains an edge uv . Recall that we assumed that A contains no red edges, thus uv is a blue edge. Let $w \in A \setminus \{u, v\}$. Consider the graph $H = G[S', T]$ where $S' = S \setminus \{w\}$. We claim that H_B contains a blue Hamilton path with ends u, v . Indeed, let H' be the graph obtained from $G_B[S', T]$ by adding a vertex z to T which is connected only to u, v . Clearly, H contains a Hamilton path with ends u, v if and only if H' is Hamiltonian. As in the previous argument, it follows from Theorem 7.1 that H' is Hamiltonian.

From now on, we assume that $k \geq 3$, so $|S| \geq n/2 + 1$. Denote

$$A_i = N_R(y, S_i) \text{ and } B_i = N_R(y, T_i). \quad (17)$$

Recall that $\deg_B(y) \leq n/16$. It follows that at most one of the sets A_1, A_2, B_1, B_2 can have size at most $n/16$.

Case 2.1: $e(G_B[S_1, T_2]) \geq 10\eta n^2$

Suppose first that $|A_2| \geq n/16$. This case can be treated similarly to previous cases where $X = \emptyset$. If $|S_1| > |T_1|$, we may continue similarly to Condition (2) from Case 1.1. If $|S_1| = |T_1|$ we continue similarly to Case 1.2 and if $|S_1| < |T_1|$, we continue as in Case 1.3. The only difference in the arguments is that when considering the graph $G[S'_2 \cup T'_2]$ we partition it into a blue path with suitable ends and a red path contained in $A_2 = N_R(y, S_2)$ rather than a red cycle (by the lower bound on A_2 , we have that $G[A_2]$ is almost complete, hence we may indeed do so).

We now suppose that $|A_2| \leq n/16$. This implies that $|A_1|, |B_1|, |B_2| \geq n/16$. Suppose first that $G_B[S_1, S_2]$ contains a path of length $10\eta n$. We may pick vertex disjoint paths $P_1 \subseteq G_B[S_1, S_2]$ and $P_2 \subseteq G_B[S_1, T_2]$ of length at most $10\eta n$ such that the following assertions hold.

- P_1 has ends $s_1 \in S_1, s_2 \in S_2$ and P_2 has ends $s_3 \in S_1, t_3 \in T_2$.
- $|S_i \setminus V(P_1)| < |T_i|$ for $i \in [2]$.
- Denote by U the set of the inner vertices from the paths P_1, P_2 and $S'_i = S_i \setminus U, T'_i = T_i \setminus U$. Then $|S'_1| = |T'_1|$ (and thus $|S'_2| < |T'_2|$).

Indeed, to construct the paths P_1, P_2 , first pick $P_1 \in G_B[S_1, S_2]$ to be long enough so that the second condition holds. Then by Theorem 7.3, the graph $G[S_1, T_2] \setminus V(P_1)$ contains a path of length $10\eta n$ (recall that we assume $e(G_B[S_1, T_2]) \geq 10\eta n^2$), and we may take P_2 to be a subpath satisfying the third condition. We continue as usual, by finding a Hamilton path in $G_B[S'_2, T'_2]$ with ends s_2, t_3 and partitioning $G[S'_1, T'_1]$ into a blue path with ends s_1, s_3 and a red path contained in B_1 .

We now suppose that $G_B[S_1, S_2]$ does not have a path of length $10\eta n$, so $G_B[S_1, S_2]$ is almost complete. It follows that we may pick $u, v \in A_1$ such that the set $D = N_R(u, S_2) \cap N_R(v, S_2)$ has size at least $n/8$. Define $S'_1 = S_1 \setminus \{u, v\}$ and $S' = S \setminus \{u, v\}$. We shall now continue as before, partitioning the graph $G[S' \cup T]$ into a blue cycle and a red path with ends in D .

If $|S'_1| > |T_1|$, we continue as in condition (2) of Case 1.1, picking say two paths in $G_B[S'_1, T_2]$ with one end in S'_1 and the other in T_2 , whose removal from $G_B[S'_1, T_1]$ leaves a balanced graph. Again, the difference is that when we consider the remainder of the graph $G[S_2 \cup T_2]$, we partition it into a blue path with suitable ends and a red path contained in D . Note that in order for this path to complete the path $(u y v)$ into a cycle, it has to contain at least one vertex. Recall that $|S| \geq |T| + 3$ so this is indeed possible.

If $|S'_1| = |T_1|$, we proceed similarly. It is easy to check that $G_B[S_2, T_1]$ contains an edge, since either $|S_2| \geq n/4 - 1/2$ or $|S'_1| = |T_1| \geq n/4 - 1/2$. Thus we may pick one edge from $G_B[S'_1, T_2]$ and another from $G_B[S_2, T_1]$ and continue as before.

Finally, we need to consider the case $|S'_1| < |T_1|$. Here we can follow the argument of Case 1.3. Note that we need to ensure that a version of Claim 16.4 holds for the graph $G[S', T]$. Indeed, take $H = G_B[S'', T_1]$, where S'' is obtained from S' by removing a vertex from S'_1 . We claim that H has a matching saturating T_1 . Indeed, let $W \subseteq T_1$. We consider four ranges for the size of W .

- $|W| = 1$. Clearly, $|N_H(W)| \geq |W|$.

- $2 \leq |W| \leq n/4 - 2$. Since any vertex of W other than x has at most one red neighbour in $G[S'', T]$, we have

$$|N_H(W)| \geq |S''| - n/4 = |S| - n/4 - 3 \geq n/4 - 2 \geq |W|.$$

- $n/4 - 2 < |W| \leq n/4 - 1$. There exists a vertex in W which has no red neighbours in $G[S'', T]$, thus $|N_H(W)| \geq n/4 - 1 \geq |W|$.
- $|W| > n/4 - 1$. Many vertices of S'' have at least one neighbour in W , implying that $|N_H(W)| > |T_1| \geq |W|$.

Thus we may pick a matching M_1 from in $G_B[S_2, T_1]$ of size $|T_1| - |S'_1| + 1$. Repeating almost the same argument, we deduce that the graph H' obtained from H by removing all edges spanned by $V(M)$, has a matching saturating T_1 . From here we may continue as in Claim 16.4 to obtain paths Q_1, Q_2 . Finally, as explained before, when considering the graph remaining from $G[S_2, T_2]$, we partition it into a blue path with suitable ends and a red path, on at least one vertex, contained in D .

From now on we may assume that $e(G_B[S_1, T_2]) \leq 10\eta n^2$, so S_1, S_2 become practically interchangeable.

Case 2.2: $G_B[S_1, S_2]$ has a path of length $20\eta n$

Recall that $A_i = N_R(y, S_i)$ and $B_i = N_R(y, T_i)$.

If $|S_1| > |T_1|$ and $|A_2|, |B_2| \geq n/16$, we continue as in Condition (3) in Case 1.1. The difference is that when considering the graph remaining from $G[S_2 \cup T_2]$ we partition it into a blue path with the given ends and a red path (which may be empty) contained in either A_2 or B_2 . If $|S_1| = |T_1|$ and $|A_2|, |B_2| \geq n/16$, it is easy to check that $G_B[S_2, T_1]$ is non empty, so we may proceed as before. The case $|S_2| \geq |T_2|$ and $|A_1|, |B_1| \geq n/16$ follows analogously.

Since at most one of the sets A_1, A_2, B_1, B_2 has size at most $n/16$, it remains to consider the case where $|S_1| < |T_1|$ or $|S_2| < |T_2|$. Without loss of generality, $|S_1| < |T_1|$, so $|S_2| > |T_2|$. If $|A_1|, |B_1| \geq n/16$, we are done. Thus we may assume $|A_2| \geq n/16$. But then we may proceed as in Case 1.3, to partition $V(G)$ into a blue cycle and a red path contained in A_2 .

From now on, we may assume that $G_B[S_1, S_2]$ has no path of length $20\eta n$, implying that $G_R[S_1, S_2]$ is almost complete. Also, without loss of generality,

$$|A_2| \geq n/16.$$

Case 2.3: $|S_1| \geq |T_1| + 1$ and $|S_2| \geq |T_2| + 3$

It is easy to check that in this case $G[S_1, T_2] \cup G[S_2, T_1]$ contains a blue matching of size 2. Denote the matching by $\{e_1, e_2\}$ and without loss of generality suppose e_i has ends $s_i \in S_2, t_i \in T_1$. If $G[S_1]$ has a blue path of length $20\eta n$, the graph $G_B[S_1 \cup T_1]$ has a Hamilton path with ends t_1, t_2 . We proceed as before, to partition $G[S_2 \cup T_2]$ into a blue path with ends s_1, s_2 and a red path contained in A_2 .

We may thus assume that $G_R[S_1]$ is almost complete. If $e(G_B[S_2]) \geq 10\eta n^2$, pick $u, v \in A_2$ such that the set $D = N_R(u, S_1) \cap N_R(v, S_1)$ has size at least $n/8$. Denote $S'_2 = S_2 \setminus \{u, v\}$. It is easy to verify that $G_B[S'_2, T_2]$ has a Hamilton path with ends s_1, s_2 . We form a red cycle by picking a red path P_2 in D of length $|S_1| - |T_1|$ and joining it to (u, v) (note that this is indeed a cycle since $|S_1| - |T_1| \geq 1$). The graph $G_B[S_1 \setminus V(P_2), T_1]$ has a Hamilton path with ends t_1, t_2 , completing the required partition.

Finally, if $e(G_B[S_2]) \leq 10\eta n^2$, we have that $G_R[S]$ is almost complete, hence we may pick a red cycle C in $S \cup \{y\}$ with the following properties.

- $y \in V(C)$.
- $|V(C) \cap S_1| = |S_1| - |T_1| + 1$.
- $|V(C) \cap S_2| = |S_2| - |T_2| - 1$.

It is easy to verify that the graph $G_B \setminus V(C)$ is Hamiltonian.

Case 2.4: $|S_1| = |T_1|$, $|S_2| = |T_2| + 2$ or $|S_2| = |T_2| + 1$

These cases may be dealt with similarly to the previous one. Denote $H_1 = G_B[S_1, T_2]$, $H_2 = G_B[S_2, T_1]$ and $H = H_1 \cup H_2$.

If $|S_1| = |T_1|$, if we may pick a 2-matching in H with at least one edge from H_2 , we may continue as in the previous case. Namely, we need to show that H_2 is non-empty. This follows if $|S_2| > n/4 - 1$ or $|T_1| > n/4 - 1$, so we may assume that $|S_2|, |T_1| \leq n/4 - 1$, implying that $|S| = |S_1| + |S_2| = |T_1| + |S_2| \leq n/2 - 2$, a contradiction.

If $|S_2| = |T_2| + 2$ (so $|S_1| \geq |T_1| + 1$) we need to show that H_1 is non-empty. If it is empty, we have $|S_1|, |T_2| \leq n/4 - 1$, implying that $|S| \leq n/2$, a contradiction.

If $|S_2| = |T_2| + 1$, we need to show that H_1 contains a 2-matching. If not, we have $|S_1|, |T_2| \leq n/4$, and in addition either $|S_1| \leq n/4 - 1$ or $|T_2| \leq n/4 - 1$. In particular, $|S_1| + |T_2| \leq n/2 - 1$, hence $|S| \leq n/2$, a contradiction.

Case 2.5: $|S_1| < |T_1|$

We proceed as in Case 1.3, partitioning the graph remaining from $G[S_2 \cup T_2]$ into a blue path with suitable ends and a red path contained in A_2 .

Case 2.6: $|S_2| \leq |T_2|$

We may continue as in the last part of Case 2.1. We pick $u, v \in A_2$ such that $|N_R(u, v) \cap S_1| \geq n/8$ and proceed to partition $G[S' \cup T]$ (where $S' = S \setminus \{u, v\}$) into a blue cycle and a red path contained in the red neighbourhood of u, v in S_1 , using an analogue of Claim 16.4.

□

The proof of Lemma 7.14 concludes the proof of our main Theorem, Theorem 1.1. We finish this paper with some concluding remarks.

17 Concluding Remarks

More than two colours - results and future work

As a further line of research, one may consider colourings of K_n with more than two colours. Gyárfás [13] conjectured that for every r -colouring of K_n , the vertex set may be partitioned into at most r monochromatic paths. Erdős, Gyárfás and Pyber [11] considered partitions into monochromatic cycles rather than paths. They defined $c(r)$ to be the smallest t such that whenever a complete graph is r -coloured, it may be partitioned into $c(r)$ monochromatic cycles. They proved that $c(r)$ is bounded, and furthermore $c(r) \leq cr^2 \log r$ for some constant c . Note that Bessy and Thomassé's result [5], mentioned in the introduction, implies that $c(2) = 2$.

Gyárfás, Ruszinkó, Sárközy and Szemerédi [15] proved that $c(r) \leq cr \log r$, which is the best known upper bound on $c(r)$ so far. The same authors [16] proved an approximate result of the last conjecture for $r = 3$. Furthermore, they showed that for large enough n , if K_n is 3-coloured, it may be partitioned into 17 monochromatic cycles. However, it turns out that the full conjecture is false, even for $r = 3$, as shown by Pokrovskiy [23]. Nevertheless, in the same paper, he proved Gyárfás' conjecture for $r = 3$, so it may still be the case that Gyárfás' conjecture holds in general. In addition, the counter examples given in [23] are 3-colourings of K_n for which all but one vertex may be covered by vertex-disjoint monochromatic paths. This raises the following question: is it true that for every r -colouring of K_n all but at most $c = c(r)$ vertices may be covered by r vertex-disjoint monochromatic paths?

Finally, it is natural to consider the Schelp-type version of these problems, namely for graphs with large minimum degree rather than for complete graph. An example for a concrete question of this type is: what is the smallest value of c such that any 3-coloured graph G on n vertices and minimum degree $\delta(G) \geq cn$ can be partitioned into three monochromatic paths?

We believe that the methods we have employed for the proof of Theorem 1.1 may prove useful in resolving the latter questions, as well as many others, regarding partitions of r -coloured graphs into paths and cycles.

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